J. Differential Equations 249 (2010) 965–988

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On the compactness problem of extremal functions to sharp Riemannian L^p-Sobolev inequalities

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ARTICLE INFO

Article history:
Received 11 December 2009
Available online 27 March 2010

MSC:
32Q10
53C21

Keywords:
Sharp Sobolev inequalities
Extremal functions
Compactness problem

ABSTRACT

Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 2$. For $1 < p < q_0 = \min\{2, \sqrt{n}\}$, Djadli and Druet (2001) [13] proved the existence of extremal functions to the following sharp Riemannian L^p-Sobolev inequality:

$$\|u\|^p_{L^{p^*}(M)} \leqslant K(n,p)^p \|\nabla u\|^p_{L^p(M)} + B_0(p,g)\|u\|^p_{L^p(M)},$$

where $p^* = \frac{np}{n-p}$ and $K(n,p)^p$ and $B_0(p,g)$ stands for, respectively, the first and second Sobolev best constants for this inequality. Let then $E_g(p)$ be the corresponding extremal set normalized by the unity L^{p^*}-norm. In contrast what happens in the whole space $\mathbb{R}^n$ for $1 < p < n$ and in the Euclidean sphere $\mathbb{S}^n$ for $p = 2$, we establish the C^0-compactness of $E_g(p)$ for any $1 < p < q_0$. Moreover, we address the question from a uniform viewpoint on p. Precisely, we prove that the set $\bigcup_{1+\varepsilon \leqslant p \leqslant q_0-\varepsilon} E_g(p)$ is C^0-compact for any $\varepsilon > 0$. The continuity of the map $p \in [1, q_0) \mapsto B_0(p,g)$ is discussed in detail since it plays a key role in the proof of the main theorem.

1. An overview and the main theorem

A lot of attention has been paid to so-called sharp Sobolev type inequalities, very often in connection with concrete problems from geometry and physics (cf. Aubin [1,2], Beckner [5], Brezis and Nirenberg [6], Carlen and Loss [8], Carleson and Chang [9], Escobar [18], Hebey and Vaugon [23],

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.02.004

Lieb [29], Lieb and Thirring [30], Moser [33], Struwe [37], Talenti [38], Trudinger [40], among others). Particularly, considerable work has been devoted to the study of extremal functions to these inequalities in recent decades. Such functions are connected, for instance, with the computation of ground state energy in some physical models and with isoperimetric inequalities (cf. Aubin [1], Brouttelande [7], Collion, Hebey and Vaugon [11], Demyanov and Nazarov [12], Djadli and Druet [13], Druet [15], Druet, Hebey and Vaugon [17], Hebey [20,21], Humbert [24], Yuxiang Li [26], Zhu [41]).

The goal of the present paper is to discuss the compactness of extremal functions to sharp Riemannian L^p-Sobolev inequalities on smooth compact Riemannian manifolds without boundary. Before we go further and exhibit our target problem, a little bit of notation and overview should be presented.

For $n \geqslant 2$, $1 \leqslant p < n$ and $p^* = \frac{np}{n-p}$, the Euclidean L^p-Sobolev best constant is given by

$$K(n,p)^p := \sup\left\{ \frac{\|u\|^p_{L^{p^*}(\mathbb{R}^n)}}{\|\nabla u\|^p_{L^p(\mathbb{R}^n)}} \colon u \in L^{p^*}\big(\mathbb{R}^n\big) \setminus \{0\},\ |\nabla u| \in L^p\big(\mathbb{R}^n\big) \right\}.$$

Independently, Aubin [1] and Talenti [38] showed that

$$K(n,p)^p = \frac{1}{n^{1+\frac{p}{n}}} \left(\frac{p-1}{n-p}\right)^{p-1} \left(\frac{\Gamma(n+1)}{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})\omega_n}\right)^{\frac{p}{n}}$$

for $1 < p < n$, and

$$K(n,1) = \lim_{p \to 1^+} K(n,p)^p = n^{-1} \omega_n^{-\frac{1}{n}},$$

where ω_n and Γ denote, respectively, the volume of the unit Euclidean n-ball and the usual Gamma function. Moreover, they also showed that the supremum is attained and that, for $1 < p < n$, the corresponding maximizers are explicitly given, modulo nonzero constant multiple, by

$$u_{\lambda, x_0}(x) = \lambda^{\frac{n}{p^*}} v_0\big(\lambda(x - x_0)\big),$$

where $\lambda > 0$, $x_0 \in \mathbb{R}^n$ and

$$v_0(x) = \Big(1 + \big(n(n-p)\big)^{-1} K(n,p)^{-p} |x|^{\frac{p}{p-1}}\Big)^{-\frac{n}{p^*}}.$$

The function v_0 is characterized as the unique solution of the equation

$$-\Delta_p v = K(n,p)^{-p} v^{p^*-1} \quad \text{in } \mathbb{R}^n,$$

where Δ_p denotes the Euclidean p-Laplace operator, satisfying $v_0 \in D^{1,p}(\mathbb{R}^n)$, $0 < v_0 \leqslant 1$, $v_0(0) = 1$ and $\|v_0\|_{p^*} = 1$. In other words, the set of extremal functions to the sharp Euclidean L^p-Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{p}{p^*}} \leqslant K(n,p)^p \int_{\mathbb{R}^n} |\nabla u|^p\, dx,$$

normalized by the unity L^{p^*}-norm, is given exactly by $E(p) := \{\pm u_{\lambda, x_0} \colon \lambda > 0,\ x_0 \in \mathbb{R}^n\}$.