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On the Cauchy problem for the fast diffusion equation

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Abstract

For $u_0, \frac{1}{u_0} \in L^1_{\text{loc}}(R^N)$, the author studies the existence of a kind of weak solution to the Cauchy problem

$$u_t = \operatorname{div}\big(u^{m-1} Du\big), \quad \text{in } R^N \times (0, T],$$
$$u(x,0) = u_0(x) \geqslant 0, \quad \text{in } R^N,$$

where $m < 0$ is a constant. The uniqueness and regularity of solutions are also discussed.

MSC: 35D05; 35D10; 35K15; 35K65

Keywords: Existence; Regularity; Cauchy problem; Fast diffusion

1. Introduction

The Cauchy problem

$$u_t = \operatorname{div}\big(u^{m-1} Du\big), \quad \text{in } S_T, \tag{1.1}$$
$$u(x,0) = u_0(x) \geqslant 0, \quad \text{in } R^N, \tag{1.2}$$

has been studied by many authors, where $S_T = R^N \times (0, T]$ and m is a constant.

The equation is referred as porous medium equation for $m > 1$ and the fast-diffusion for $0 < m < 1$. The problem for these cases has been studied by several authors; we refer readers to [1,2,4,11,15] and the references therein.

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For $m < 0$, the equation is referred as the very fast diffusion. When $m \leqslant 0$, problem (1.1)–(1.2) arises naturally in certain physical applications. See [7,13,14] and their references.

To our knowledge, there are only a few papers working on such problem. In [12,17], the authors found out a necessary condition for the existence of nonnegative weak solutions. In [13], distributional solutions were shown to exist under conditions on the initial value u_0 involving a sequence of Green's functions of the balls B_R of R^N. In [5], Guy Bernard established the following existence result:

Theorem A. *Let $m < 1$. Assume that the initial value $u_0 \in C^{2,\alpha}(R^N)$ for some number $0 < \alpha \leqslant 1$ and satisfies the growth and decay conditions*

$$\frac{C_1}{(1+|x|^2)^{(1-\epsilon)/(1-m)}} \leqslant u_0(x) \leqslant C_2\big(1+|x|^2\big)^{(1-\epsilon)/(1-m)}, \quad \textit{for all } x \in R^N,$$

where $C_1 > 0$, $C_2 > 0$, $0 < \epsilon < 1$ are constants.

Then, there exists a classical solution of (1.1)–(1.2) for any $T > 0$.

For $m < 0$, by the result of [12], we know that in a sense the growth restriction in Theorem A is sharp.

In this paper we study the existence, uniqueness and regularity of weak solutions for the case $m < 0$.

Motivated by the idea in [3], we demonstrate the existence of a kind of weak solutions, which is defined as the following

Definition 1.1. A measurable function $u(x,t)$ defined in S_T is called a weak solution of (1.1)–(1.2) if, for every bounded open set Ω with smooth boundary $\partial\Omega$, setting $\Omega_T = \Omega \times (0,T)$, for all $0 \leqslant t_0 < t < T$, the following conditions are satisfied:

$$u \in C_{\mathrm{loc}}\big(0,T; L^1_{\mathrm{loc}}\big(R^N\big)\big), \qquad u^m \in L^2_{\mathrm{loc}}\big(0,T; W^{1,2}_{\mathrm{loc}}\big(R^N\big)\big), \tag{1.3}$$

and that

$$\int_\Omega u(\cdot,t)\phi(\cdot,t)\,dx\,d\tau + \int_0^t\int_\Omega \left(-u\phi_t + D\left(\frac{u^m}{m}\right)\cdot D\phi\right)dx\,d\tau = \int_\Omega u(\cdot,t_0)\phi(\cdot,t_0)\,dx\,d\tau, \tag{1.4}$$

for all testing functions $\phi \in C^1(\overline{\Omega}_T)$ with $\phi = 0$ near $\partial\Omega \times (0,T)$, as well as that

$$\lim_{t\to 0}\int_{B_R} \big|u(x,t) - u_0(x)\big|\,dx = 0, \quad \forall R > 0. \tag{1.5}$$

Here and throughout this paper B_R denotes the ball of radius R centered at the origin.