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Radial ground states and singular ground states for a spatial-dependent p-Laplace equation

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ABSTRACT

We consider the following equation

$$\Delta_p u(\mathbf{x}) + f\big(u, |\mathbf{x}|\big) = 0,$$

where $\mathbf{x} \in \mathbb{R}^n$, $n > p > 1$, and we assume that f is negative for $|u|$ small and $\lim_{u\to+\infty} \frac{f(u,0)}{u|u|^{q-2}} = a_0 > 0$ where $p_* = \frac{p(n-1)}{n-p} < q < p^* = \frac{np}{n-p}$, so $f(u,0)$ is subcritical and superlinear at infinity.
In this paper we generalize the results obtained in a previous paper, [11], where the prototypical nonlinearity

$$f(u,r) = -k_1(r)u|u|^{q_1-2} + k_2(r)u|u|^{q_2-2}$$

is considered, with the further restriction $1 < p \leqslant 2$ and $q_1 > 2$. We manage to prove the existence of a radial ground state, for more generic functions $f(u,|\mathbf{x}|)$ and also in the case $p > 2$ and $1 < q_1 < 2$. We also prove the existence of uncountably many radial singular ground states under very weak hypotheses.
The proofs combine an energy analysis and a shooting method. We also make use of Wazewski's principle to overcome some difficulties deriving from the lack of regularity.

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1. Introduction

The aim of this paper, along with [11], is to investigate positive radial solutions for equation of this type

$$\operatorname{div}\big(\nabla u|\nabla u|^{p-2}\big)+f\big(u,|x|\big)=0, \tag{1.1}$$

where $\mathbf{x}\in\mathbb{R}^n$, and $f(u,|\mathbf{x}|)$ is negative as $u\to 0$ and positive and subcritical with respect to the Sobolev critical exponent as $u\to\infty$.

Since we just consider radially symmetric solutions we will in fact study the following singular ODE where we have set $r=|\mathbf{x}|$:

$$\big(u'|u'|^{p-2}\big)'+\frac{n-1}{r}u'|u'|^{p-2}+f(u,r)=0. \tag{1.2}$$

Here and later we denote by $'$ the derivative with respect to r. The prototypical nonlinearity f we are considering is

$$f(u,r)=-k_1(r)u|u|^{q_1-2}+k_2(r)u|u|^{q_2-2}, \tag{1.3}$$

where k_1 and k_2 are positive functions which are locally Lipschitz continuous and $q_1<q_2<p^*$, where p^* is the Sobolev critical exponent. We recall that p^* is usually defined just when $n>p$ and we have $p^*=\frac{np}{n-p}$; when $n\leqslant p$ we set $p^*=\infty$. We use the following notation: we call classic a solution of (1.2) satisfying

$$u(0)=d>0 \quad\text{and}\quad u'(0)=0, \tag{1.4}$$

and sometimes we denote by $u(d,r)$ such a solution to stress the dependence on the initial condition; we call singular a solution $u(r)$ such that $\lim_{r\to 0}u(r)=\infty$.

In particular we focus our attention on the problem of existence of ground states (G.S.), of singular ground states (S.G.S.) and of crossing solutions. By G.S. we mean a nonnegative classic solution $u(\mathbf{x})$ defined in the whole of $\mathbb{R}^n$ such that $\lim_{|\mathbf{x}|\to\infty}u(\mathbf{x})=0$. An S.G.S. of Eq. (1.1) is a singular nonnegative solution $v(\mathbf{x})$ such that

$$\lim_{|\mathbf{x}|\to 0}v(\mathbf{x})=+\infty \quad\text{and}\quad \lim_{|\mathbf{x}|\to+\infty}v(\mathbf{x})=0.$$

Crossing solutions are radial solutions $u(r)$ such that $u(r)>0$ for any $0\leqslant r<R$ and $u(R)=0$ for some $R>0$, so they can be considered as solutions of the Dirichlet problem in the ball of radius R. Here and later we write $u(r)$ for $u(\mathbf{x})$ when $|\mathbf{x}|=r$ and u is radially symmetric.

In our equation an important role is played also by the critical value p_*, which is the largest q such that the trace operator $\gamma: W^{1,p}(\Omega)\to L^q(\partial\Omega)$ is continuous; i.e. $p_*:=\frac{p(n-1)}{n-p}$ when $n>p$; when $n\leqslant p$ we set $p_*=\infty$. We will always assume the following:

$$\text{F0}\quad\begin{cases}\bullet\ \text{The function } f(u,r) \text{ is continuous in } \mathbb{R}^2, \text{ Lipschitz continuous in}\\ \quad\text{both the variables for } u,r>0.\\ \bullet\ f(-u,r)=-f(u,r) \text{ for any } r\geqslant 0 \text{ and for any } u\in\mathbb{R}.\\ \bullet\ \text{There are } \nu>0 \text{ and } p<q<p^* \text{ such that, for any } 0\leqslant r\leqslant\nu\\ \quad\lim_{u\to\infty}\frac{f(u,r)}{|u|^{q-1}}=a_0(r)>0 \text{ and } a_0(r) \text{ is continuous.}\end{cases}$$

We have implicitly assumed that $\lim_{r\to 0}f(u,r)$ is bounded. In fact this hypothesis is not really restrictive since, even when $\lim_{r\to 0}f(u,r)=+\infty$ for any $u>0$, usually it is possible to reduce the