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The period function of classical Liénard equations

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Dedicated to Carmen Chicone on the occasion of his 60th birthday

Abstract

In this paper we study the number of critical points that the period function of a center of a classical Liénard equation can have. Centers of classical Liénard equations are related to scalar differential equations $\ddot{x} + x + f(x)\dot{x} = 0$, with f an odd polynomial, let us say of degree $2\ell - 1$. We show that the existence of a finite upperbound on the number of critical periods, only depending on the value of ℓ, can be reduced to the study of slow–fast Liénard equations close to their limiting layer equations. We show that near the central system of degree $2\ell - 1$ the number of critical periods is at most $2\ell - 2$. We show the occurrence of slow–fast Liénard systems exhibiting $2\ell - 2$ critical periods, elucidating a qualitative process behind the occurrence of critical periods. It all provides evidence for conjecturing that $2\ell - 2$ is a sharp upperbound on the number of critical periods. We also show that the number of critical periods, multiplicity taken into account, is always even.

MSC: 34C25; 34C26; 34E15; 34E10

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1. Introduction

This paper deals with the period function of (classical) polynomial Liénard equations of center type. The period $T(h)$ of periodic orbits of a vector field, parametrized by a coordinate h on a

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transverse section, may have one or more critical points. It is well known (see [1]) that any given polynomial vector field of center type has a finite number of critical periods on a period annulus contained in a compact region. The method of proof in [1] is to show that the derivative of the period tends to $+\infty$ as the periodic orbits approach a boundary of the annulus of periodic orbits which contains both regular and singular points. This boundary may be chosen on the compactification of the phase space, i.e. a boundary at infinity is permitted in the proof under rather mild conditions. In the case of the classical Liénard equations it is quite easy, based on the techniques presented in [1], to prove that individual systems have a finite number of critical periods.

A natural question that comes up is whether or not there is a *uniform* bound on the number of critical points of the period function, depending only on the degree of the polynomial vector field. This question relates to the result in [1] much like the Hilbert 16th problem relates to Dulac's theorem concerning the finiteness of the number of limit cycles of polynomial vector fields.

Our choice to study this kind of finiteness problem first inside the class of classical polynomial Liénard equations can be well motivated: there is only one singular point, we can express in a straightforward way that it is a center, the behavior of these equations at infinity is well-studied (see [2]), and last but not least: there is an elegant way of compactifying the space of polynomial Liénard equations of given degree; we refer to [3], but similar ideas can be found in [4]. Together with a compactification of the phase plane (by means of a Poincaré–Lyapunov compactification) the above-mentioned finiteness problem can now be dealt with systematically.

Essentially, the space of Liénard equations is compactified by adding singular perturbation problems to them. In view of the finiteness problem of critical periods, we will see that precisely the study of these "slow–fast Liénard equations" will prove to be representative.

The finiteness of the number of critical periods of *local* periodic orbits perturbing from the origin has already been studied in [5]: if $h = 0$ describes the origin, then the order of zero of the Taylor development of $T'(h)$ at $h = 0$ can be used to bound the number of critical periods. We would like to stress that the results in our paper are global in the sense that the entire annulus of periodic orbits is considered.

Periodic orbits of slow–fast Liénard equations are close to so-called limit periodic sets. Such limit periodic sets are formed by glueing together slow trajectories and fast orbits (see Section 3). We will see that, generically, the derivative of the period function is related to the way these limit periodic sets change in shape. Using this point of view, we are able to give a clear geometric proof for a lower bound on the number of critical periods.

In a way, in the opposite of the boundary provided by singular slow–fast Liénard systems, we find perturbations of the so-called *central system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - yx^n, \end{cases}$$

n odd (see Section 2 for an explanation). Following the ideas of Chicone and Jacobs in [5], we study local critical periods of such perturbations by calculating the Taylor expansion of the period function at the origin, and we prove that all critical periods are visible locally i.e. no other critical periods appear outside the origin. The upperbound that follows from this study coincides with the lower bound that was obtained using singular perturbation theory. In [5], some criteria are formulated to show the exactness of the presented upper bound. Unfortunately, these criteria are not satisfied for perturbations of the central system within the given class of vector fields. This is essentially, but not solely, the consequence of the lack of a sufficient number of parameters. We