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Subsonic solutions to compressible transonic potential problems for isothermal self-similar flows and steady flows ☆

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Abstract

We establish boundary and interior gradient estimates, and show that no supersonic bubble appears inside of a subsonic region for transonic potential flows for both self-similar isothermal and steady problems. We establish an existence result for the self-similar isothermal problem, and improve the Hopf maximum principle to show that the flow is strictly elliptic inside of the subsonic region for the steady problem.

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1. Introduction

For Riemann problems in compressible two-dimensional self-similar flow, the governing equation becomes quasilinear and changes its type. Namely, the type of the flow in the far-field is hyperbolic and near the origin it is mixed. A natural question is whether a super sonic bubble can appear inside of a subsonic region. This question was discussed in [17] for isothermal irrotational self-similar flow. It was also pointed out in [17] that for steady isentropic irrotational potential flow, Ringleb's example [13] shows a possibility that the nonsmooth flow can have regions with

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both supersonic and subsonic. The main goal of this manuscript is to answer this question for steady flow and self-similar isothermal flow.

In this manuscript, for the domain satisfying a uniform exterior sphere condition and under certain conditions on the boundary data, we establish boundary gradient and interior gradient estimates which are crucial in verifying the type of the flow, and show that there is no supersonic bubble inside of a subsonic region for isothermal self-similar flow and steady flow. For isothermal self-similar flow, we establish the existence result. For the steady problem, since the problem becomes homogeneous, by improving the Hopf principle, we show that a smooth solution is either subsonic (strictly elliptic) everywhere inside of the region or sonic everywhere.

We note that an explicit solution for Ringleb's example [13], obtained by hodograph transformations for the steady problem, indicates that there exists a sonic line which forms *cusps*, and the flow is first subsonic, then supersonic, and finally subsonic again when it passes near the cusps. Typically, when the boundary forms cusps, the solution loses its smoothness near the cusps. Hence our results do not apply to the example. This implies the possibility that the hypotheses in our results are not only sufficient but in some sense *necessary* to prevent the supersonic bubble inside the subsonic region. We discuss Ringleb's example more in details later in this section.

In earlier work [19], we showed that no supersonic bubble occurs internally to a subsonic region for isentropic self-similar potential flow in two space dimensions. The techniques presented in [19] rely on the fact that the adiabatic exponent γ satisfies $\gamma > 1$ and a structure of the governing equation, and thus it is not immediate that self-similar isothermal flow, that is the sound speed becomes constant ($\gamma = 1$), and steady flow may hold the same result. Furthermore in [19] the boundary gradient estimates were established for a convex domain. Closely related with this work, Elling and Liu [14] presented a partial result of the strict ellipticity of the potential flow under the assumption that the ellipticity in the subsonic region is given. Chen and Feldman [11] announced that they established an existence result for the potential flow for the regular weak shock reflection by a nearly vertical wedge.

We first give a brief summary of the self-similar potential flow. Details can be found in [12,21] and the references therein. From the unsteady potential flow equation in two space dimension, where $(u, v) = \nabla\Phi$ is the velocity and ρ is the density, equations of conservation of mass and Bernoulli's law are written as

$$\rho_t + \operatorname{div}(\rho\nabla\Phi) = 0, \tag{1}$$

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + i(\rho) = \text{const}, \tag{2}$$

where $di(\rho)/d\rho = c^2(\rho)/\rho$ with c the sound speed. Assume that the gas is polytropic, $c^2 = \gamma\rho^{\gamma-1}$ with either $1 < \gamma < \infty$ isentropic, or $\gamma = 1$ isothermal. For simplicity we take the constant in Bernoulli's law (2) as zero. This leads to

$$\rho = \rho\left(-\Phi_t - \frac{1}{2}|\nabla\Phi|^2\right) = \left(\frac{\gamma-1}{\gamma}\left(-\Phi_t - \frac{1}{2}|\nabla\Phi|^2\right)\right)^{1/(\gamma-1)}.$$

Introducing the self-similar coordinates $\xi = x/t$ and $\eta = y/t$ and letting $\Phi(t, x, y) = t\psi(\xi, \eta)$, Eq. (1) becomes

$$\left(c^2 - (\psi_\xi - \xi)^2\right)\psi_{\xi\xi} - 2(\psi_\xi - \xi)(\psi_\eta - \eta)\psi_{\xi\eta} + \left(c^2 - (\psi_\eta - \eta)^2\right)\psi_{\eta\eta} = 0. \tag{3}$$