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On the stochastic Benjamin–Ono equation

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Abstract

We discuss the Cauchy problem for the stochastic Benjamin–Ono equation in the function class $H^s(R)$, $s > 3/2$. When there is a zero-order dissipation, we also establish the existence of an invariant measure with support in $H^2(R)$. Many authors have discussed the Cauchy problem for the deterministic Benjamin–Ono equation. But our results are new for the stochastic Benjamin–Ono equation. Our goal is to extend known results for the deterministic equation to the stochastic equation.

MSC: 35L65; 35R60; 60H15

Keywords: Benjamin–Ono equation; Cauchy problem; Brownian motion; Invariant measure

0. Introduction

In this paper we will discuss the Cauchy problem for the stochastic Benjamin–Ono equation. Our main goal is to establish the existence of a solution to the Cauchy problem and to prove the existence of an invariant measure. The Cauchy problem is formulated as follows:

$$u_t + uu_x + \mathcal{H}(u_{xx}) = \sum_{j=1}^{\infty} g_j \frac{dB_j}{dt}, \quad (t, x) \in (0, \infty) \times R, \tag{0.1}$$

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$$u(0,x)=u_0(x), \quad x\in R, \tag{0.2}$$

where $\mathcal{H}(\cdots)$ is the Hilbert transform and $g_j=g_j(t,x)$, $j=1,2,\ldots$. The right-hand side corresponds to a random noise which is white in the time variable. When $g_j\equiv 0$ for all $j\geqslant 1$, (0.1) reduces to the well-known Benjamin–Ono equation, which describes unidirectional propagation of nonlinear dispersive waves [2,17], and has been extensively studied by many authors. See [1,8,9,12,15,16,18,20,23]. However, to the author's knowledge, the Cauchy problem for the stochastic Benjamin–Ono equation (0.1) has not been investigated. Since the K-dV equation is the most well-known among all model equations which describe dispersive waves, other equations are often compared to the K-dV equation. The K-dV equation has stronger dispersion mechanism than the Benjamin–Ono equation, and the existence of a solution can be established through the variation of constants formula and the semigroup associated with the principal part of the equation. This approach has a stochastic version for stochastic evolution equations, and gives rise to a stochastic convolution when the forcing term is a white noise. Hence, for the stochastic K-dV equation, [6,19] proved the existence of a solution by careful analysis of the stochastic convolution. This approach covers a broad class of stochastic evolution equations. The monograph [4] presents a comprehensive study of the general stochastic evolution equations using this method. However [16] showed that Picard iteration scheme via the variation of constants formula fails the deterministic Benjamin–Ono equation. Thus we have to employ a different approach. By regularizing the equation and the data, we first obtain a pathwise solution which is sufficiently smooth in the space variable for each sample point. We need sufficient regularity of solutions to justify manipulations for the energy estimates. This first step is essentially the same as for the deterministic equation; see [1]. The second step is to obtain necessary stochastic a priori estimates, where integral invariants play a crucial role. This requires various new stochastic estimates. We will borrow some technical estimates for the deterministic equation from [1] which presents a comprehensive analysis of integral invariants. We also borrow some analytical tools from [18]. Then, by a measure-theoretic argument, we can obtain a desired solution for the original equation. Here we will establish an existence result in the function class H^s, $s>3/2$. We also obtain estimates of the mean energy for $s=2$. The details of proof will be presented in Section 2 below.

After the global Cauchy problem, we will prove the existence of an invariant measure when Eq. (0.1) includes an additional term of zero-order dissipation. Such a term can describe variable depth in the flow model; see [14]. An invariant measure is an important object in the study of stochastic dynamics. It corresponds to a stationary solution of a deterministic equation. If the initial datum has the probability distribution equal to an invariant measure, then the probability distribution of the evolving solution is invariant in time. There are some general results on the existence of invariant measures for stochastic evolution equations; see [4,5]. But the method of such results do not cover the stochastic Benjamin–Ono equation. Here we will use the recent result on a certain class of stochastic evolution equations [13, Theorem 1.1], where some sufficient conditions for the existence of an invariant measure are presented. We will verify those conditions. This involves various technical issues. One of the required conditions is that the time-average of the norm of a solution in the basic function class must be bounded uniformly in time. Hence, we need an extra term of zero-order dissipation, which dissipates the energy due to the ran-