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# Spectral analysis of singular ordinary differential operators with indefinite weights

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ABSTRACT

In this paper we develop a perturbation approach to investigate spectral problems for singular ordinary differential operators with indefinite weight functions. We prove a general perturbation result on the local spectral properties of selfadjoint operators in Krein spaces which differ only by finitely many dimensions from the orthogonal sum of a fundamentally reducible operator and an operator with finitely many negative squares. This result is applied to singular indefinite Sturm–Liouville operators and higher order singular ordinary differential operators with indefinite weight functions.



## 1. Introduction

Sturm–Liouville differential operators and higher order ordinary differential operators with indefinite weight functions have attracted a lot of attention in the recent past. In many situations it is possible to apply techniques from operator theory in indefinite inner product spaces and to obtain in this way information on the spectral structure of the indefinite differential operator, see, e.g., [3,6, 9–11,19–25,29,34–37,39].

Let us consider the Sturm–Liouville differential expression

$$\ell = \frac{1}{w}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right), \tag{1.1}$$

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where $w, p^{-1}$ and $q$ are real valued locally integrable functions on some bounded or unbounded interval $(a,b)$ and $p(x) > 0$, $w(x) \neq 0$ for almost every $x \in (a,b)$. It will be assumed that the weight function $w$ has different signs on subsets of $(a,b)$ of positive Lebesgue measure. In this case $\ell$ is said to be an *indefinite* Sturm–Liouville expression, and it is convenient also to consider the definite counterpart of $\ell$,

$$\tau = \frac{1}{|w|}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right). \tag{1.2}$$

The operators associated to (1.1) and (1.2) act in the weighted $L^2$-space $L^2_{|w|}(a,b)$ which consists of all (equivalence classes of) complex valued measurable functions $f$ on $(a,b)$ such that $|f|^2|w|$ is integrable. Equipped with the scalar product

$$(f,g) = \int_a^b f(x)\overline{g(x)}\big|w(x)\big|\,dx, \quad f,g \in L^2_{|w|}(a,b), \tag{1.3}$$

$L^2_{|w|}(a,b)$ is a Hilbert space and the definite Sturm–Liouville differential expression (1.2) is formally symmetric with respect to the positive definite inner product (1.3). The spectral properties of the selfadjoint realizations of $\tau$ in the Hilbert space $L^2_{|w|}(a,b)$ have been studied comprehensively, see, e.g., the monographs [47,48,50–52] as introductory texts and for further references.

In contrast to $\tau$ the indefinite Sturm–Liouville expression $\ell$ is not symmetric with respect to (1.3), but becomes symmetric with respect to the indefinite inner product

$$[f,g] = \int_a^b f(x)\overline{g(x)}w(x)\,dx, \quad f,g \in L^2_{|w|}(a,b), \tag{1.4}$$

and the challenging problem is now to investigate the spectral properties of the differential operators associated to $\ell$ which are selfadjoint with respect to (1.4). The Hilbert space scalar product (1.3) and the Krein space inner product (1.4) are connected via $[J\cdot,\cdot] = (\cdot,\cdot)$, where $J$ is the multiplication operator with the function $x \mapsto \operatorname{sgn}(w(x))$. Formally, we have $J\tau = \ell$ and hence every selfadjoint realization $A$ of $\tau$ in the Hilbert space $L^2_{|w|}(a,b)$ induces a $J$-selfadjoint realization $JA$ of $\ell$, i.e., an operator which is selfadjoint in the Krein space $(L^2_{|w|}(a,b), [\cdot,\cdot])$, and vice versa. We point out that the spectral properties of operators which are $J$-selfadjoint differ essentially from the spectral properties of selfadjoint operators in Hilbert spaces, e.g., the spectrum is in general not real and may even be empty or cover the whole complex plane.

Since the pioneering work [19] by B. Ćurgus and H. Langer in 1989 the spectral structure of the $J$-selfadjoint realizations of $\ell$ (and also of higher order ordinary differential operators with indefinite weights) in the regular case, i.e., the interval $(a,b)$ is bounded and the coefficients are integrable up to the endpoints, is well understood. Namely, since every selfadjoint realization of the regular differential expression $\tau$ in the Hilbert space $L^2_{|w|}(a,b)$ is semibounded from below and the spectrum of such a differential operator $A$ consists only of eigenvalues which accumulate to $+\infty$, it can be shown with the help of abstract perturbation arguments that the resolvent set of any $J$-selfadjoint realization of $\ell$ is nonempty, the spectrum of such a regular indefinite Sturm–Liouville operator $JA$ is discrete and the form $[JA\cdot,\cdot] = (A\cdot,\cdot)$ has finitely many negative squares. It follows that the nonreal spectrum of $JA$ consists of (at most) finitely many pairs of eigenvalues which are symmetric with respect to the real line and that the real eigenvalues accumulate to $+\infty$ and $-\infty$; cf. [19, §1] and [44,45]. Under additional assumptions similar results also hold if both endpoints are in the limit circle case.

If at least one of the endpoints of the interval $(a,b)$ is in the limit point case the situation becomes much more difficult. Let us consider the particularly interesting setting where both endpoints $a$ and $b$