Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Differential Equations 235 (2007) 85–100
Journal of Differential Equations
www.elsevier.com/locate/jde

Quantum Hamiltonians with quasi-ballistic dynamics and point spectrum

César R. de Oliveira [*,1], Roberto A. Prado [2]

Departamento de Matemática, UFSCar, São Carlos, SP 13560-970, Brazil

Received 11 July 2005; revised 2 October 2006

Available online 10 January 2007

Abstract

Consider the family of Schrödinger operators (and also its Dirac version) on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$

$$H_{\omega,S}^{W} = \Delta + \lambda F\big(S^n\omega\big) + W, \quad \omega \in \Omega,$$

where S is a transformation on (compact metric) Ω, F is a real Lipschitz function and W is a (sufficiently fast) power-decaying perturbation. Under certain conditions it is shown that $H_{\omega,S}^{W}$ presents quasi-ballistic dynamics for ω in a dense G_δ set. Applications include potentials generated by rotations of the torus with analytic condition on F, doubling map, Axiom A dynamical systems and the Anderson model. If W is a rank one perturbation, examples of $H_{\omega,S}^{W}$ with quasi-ballistic dynamics and point spectrum are also presented.

MSC: 81Q10; 47B99

1. Introduction

Quantum Hamiltonians, i.e., Schrödinger and Dirac, with potentials along dynamical systems is a very interesting subject that has been considered in the mathematics and physics literature, mainly one-dimensional discrete versions. Although not explicitly stated, it is natural to expect that the more "chaotic" the underlining dynamical system, the more singular the corresponding

* Corresponding author.
E-mail addresses: oliveira@dm.ufscar.br (C.R. de Oliveira), rap@dm.ufscar.br (R.A. Prado).
1 Partially supported by CNPq (Brazil).
2 Supported by CAPES (Brazil).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.12.017

spectrum; the extreme cases could be represented by periodic potentials on one hand, which impose absolutely continuous spectrum and ballistic dynamics (see Definition 1), and random potentials on the other hand, that lead to point spectrum and absence of transport (bounded moments of the position operator). We mention the papers [9,12–14,16,17,22,25,37] for references and additional comments on important recent results on quantum dynamics for Dirac and Schrödinger operators.

Exceptions of the above picture are known, since there are examples of one-dimensional quantum models with pure point spectrum and transport. Here we refer to the random dimer model [25] for the Schrödinger case and the random Bernoulli–Dirac operator [16,17] (with no potential correlation). The first example of (Schrödinger) operators with such "unexpected behavior" has appeared in [18, Appendix 2], what the authors have called "A Pathological Example;" the potential was the almost-Mathieu (see Application 5.5.1 ahead), which is built along irrational rotations of the circle, with a combination of suitable rational approximations for the rotation angle and a rank one perturbation.

Rotations of the circle are by far the most considered dynamical systems to generate quantum potentials [6,23,30]; their finite-valued versions [3,11,19], together with substitution dynamical system potentials (see [28,29] and references therein) are mathematical models of one-dimensional quasi-crystals with predominance of singular continuous spectrum. These dynamical systems are not "chaotic," which could be characterized by positive entropy [26] or via a more dynamical definition gathered in [20]; the paradigms of chaotic systems are the Anosov and, more generally, Axiom A systems.

Since chaotic motion mimics randomness, it is natural to conjecture that for quantum operators with suitable potentials built along Axiom A (and other chaotic) systems there is a predominance of point spectrum and absence of transport. A small step in this direction are the results of [7] about Anderson localization for potentials related to the doubling map $\theta \mapsto 2\theta$ on the circle and also hyperbolic toral automorphisms—both systems have positive entropy.

The main goal of this paper is to have a close inspection on the construction of the above mentioned "unexpected example" in [18], together with the related analysis in [22], in order to get a different view of them and so provide new examples of quantum operators with quasi-ballistic dynamics, some of them with pure point spectrum. In spite of the above conjecture, as applications we can prove that for a generic (i.e., dense G_δ) set of initial conditions of Axiom A systems, as well as of chaotic dynamical systems as defined in Devaney [20], the associated quantum operators present quasi-ballistic dynamics. We will also have something to say about the random Anderson model, that is, there is a dense G_δ set of initial conditions so that the quantum operators present quasi-ballistic dynamics; see Section 5 for details and other examples. The applications are the principal contributions of this paper. From now on we shall formulate more precisely the context we work at.

Let (Ω, d) be a compact metric space. Consider the family of bounded Schrödinger operators $H^W_{\omega,S}$ given by

$$\left(H^W_{\omega,S}\psi\right)(n) = (\Delta\psi)(n) + \lambda F\left(S^n\omega\right)\psi(n) + W(n)\psi(n), \quad \omega \in \Omega, \tag{1}$$

acting on $\psi \in \ell^2(\mathbb{N})$ (with a Dirichlet, or any other, boundary condition) or the whole lattice case $\ell^2(\mathbb{Z})$, where the Laplacian Δ is the finite difference operator

$$(\Delta\psi)(n) = \psi(n+1) + \psi(n-1),$$