Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 235 (2007) 219–261
Journal of Differential Equations
www.elsevier.com/locate/jde

Persistence of wavefronts in delayed nonlocal reaction–diffusion equations

Chunhua Ou [a,*,1], Jianhong Wu [b,2]

[a] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NF, A1C 5S7, Canada*
[b] *Laboratory for Industrial and Applied Mathematics, Department of Mathematics and Statistics, York University, Toronto, ON, M3J 1P3, Canada*

Received 7 April 2006; revised 8 December 2006

Available online 9 January 2007

Abstract

We develop a perturbation argument based on existing results on asymptotic autonomous systems and the Fredholm alternative theory that yields the persistence of traveling wavefronts for reaction–diffusion equations with nonlocal and delayed nonlinearities, when the time lag is relatively small. This persistence result holds when the nonlinearity of the corresponding ordinary reaction–diffusion system is either monostable or bistable. We then illustrate this general result using five different models from population biology, epidemiology and bio-reactors.

MSC: 35K55; 35K57; 35R10; 92D25; 92D30; 92D40

Keywords: Persistence; Delay; Reaction–diffusion equations; Traveling wavefronts; Nonmonotone nonlinearity; Delayed induced nonlocality; Monostable and bistable; Population dynamics; Bio-reactor; Hybrid system; Hyperbolic–parabolic equations

* Corresponding author.
E-mail addresses: ou@math.mun.ca (C. Ou), wujh@mathstat.yorku.ca (J. Wu).
[1] This author was supported by NSERC (Canada) Discovery Grant 204509.
[2] This author was partially supported by Canada Research Chairs Program, by a Discovery Grant of Natural Sciences and Engineering Research Council of Canada, and by Mathematics for Information Technology and Complex Systems.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.12.010

1. Introduction

Since the pioneering work of Fisher [19] and Kolmogorov, Petrowsky and Piscounov [31], traveling wave solutions for reaction–diffusion equations have been extensively investigated, and this investigation has also inspired rapid development in nonlinear analysis and nonlinear dynamical systems, see [8,17,39,55] and the vast references therein.

The simplest scalar case is the following parabolic equation

$$u_t = u_{xx} + f(u) \tag{1.1}$$

with the nonlinearity f satisfying the following conditions

$$f(0) = f(1) = 0, \qquad f'(0) \neq 0 > f'(1).$$

Two prototypes of the nonlinearity have been considered: the monostable case (or the Fisher nonlinearity) where $f(u) > 0$ for $0 < u < 1$, $f'(0) > 0$; and the bistable case (or the Huxley nonlinearity) where there is $a \in (0,1)$ such that $f(u) < 0$ for $u \in (0,a)$ and $f(u) > 0$ for $u \in (a,1)$, $f'(0) < 0$. The existence of traveling waves of Eq. (1.1) can be studied using phase plane analysis: in the monostable case, Eq. (1.1) has a family of traveling waves $u = U(x-ct)$ for all wave speed $c \geqslant c^*(f) =$ minimal speed, whereas in the bistable case, there exists a unique traveling wave solution $u = U(x-ct)$ for some constant c, see for example, [52] and [18].

The study of traveling waves for many systems of reaction–diffusion equations arising from biological and physical applications becomes more complicated due to the lack of general techniques for phase space analysis, and some other approaches such as monotone iteration schemes [4,5], homotopy arguments [55] and perturbation analysis (for large speed waves) [1,39,48] have been developed.

It has been recognized that some of the well-known existence results of traveling waves must be extended to delay reaction–diffusion equations since time lags enter the dynamical models in a very natural way due to the slow signal and biochemical processes in many biological and physical systems, but such an extension becomes highly nontrivial. A fundamental difficulty arises since the equations describing the waves are no-longer systems of ordinary differential equations, but rather functional differential equations. Nevertheless, there has been substantial progress. Notably, Schaaf [47] studied a scalar reaction–diffusion equation with a single discrete delay in both the monostable case and bistable case by using ideas from phase space analysis, and he obtained the existence of traveling wavefront(s) under quasimonotonicity condition of the delayed nonlinearity. This quasimonotonicity also allows [59] to obtain the existence of traveling wavefronts for a very general delayed reaction–diffusion system, via a monotone iteration scheme coupled with the standard upper-lower solutions technique.

The difficulty in obtaining the existence of traveling waves for systems involving both spatial diffusion and temporal time lags also arises from the recent observation (see [51]) that this interaction of time lags and spatial diffusion leads to the so-called delay induced nonlocality and the resulted models taking into account biological realities take the form of reaction–diffusion equations with nonlocal and delayed nonlinearities: individuals have not been at the same point in space at previous times. For example, So, Wu and Zou [51] adopted Smith–Thieme's approach—reduction from a structured population model—to obtain a functional differential equation for the matured population in a biological system with two age classes [49] to the case of spatial dif-