J. Differential Equations 246 (2009) 3902–3931

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On convergence of solutions to equilibria for quasilinear parabolic problems

Jan Prüss [a], Gieri Simonett [b,∗,1], Rico Zacher [a,2]

[a] *Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik, Theodor-Lieser-Strasse 5, D-06120 Halle, Germany*
[b] *Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA*

A R T I C L E   I N F O

Article history:
Received 9 July 2008
Revised 23 October 2008
Available online 9 December 2008

MSC:
34G20
35K55
35B35
37D10
35R35

Keywords:
Quasilinear parabolic equations
Normally stable
Center manifolds
Nonlinear boundary conditions
Free boundary problems
Travelling waves

A B S T R A C T

We show convergence of solutions to equilibria for quasilinear parabolic evolution equations in situations where the set of equilibria is non-discrete, but forms a finite-dimensional C^1-manifold which is normally hyperbolic. Our results do not depend on the presence of an appropriate Lyapunov functional as in the Łojasiewicz–Simon approach, but are of local nature.

1. Introduction

The principle of linearized stability is a well-known and powerful tool for proving stability or instability of equilibria of nonlinear evolution equations. It is known to be true for large classes of nonlinear evolution equations, even for such which are nonlocal. The literature on this subject is

∗ Corresponding author.
E-mail addresses: jan.pruess@mathematik.uni-halle.de (J. Prüss), gieri.simonett@vanderbilt.edu (G. Simonett), rico.zacher@mathematik.uni-halle.de (R. Zacher).
[1] Partially supported by NSF, Grant DMS-0600870.
[2] Partially supported by the Deutsche Forschungsgemeinschaft (DFG), Bonn, Germany.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.10.034

large. Since here we are mainly interested in quasilinear parabolic problems, we only refer to the monograph by Lunardi [21], and to [1,24].

In this paper we will consider the following situation: suppose that for a nonlinear evolution equation we have a C^1-*manifold of equilibria* $\mathcal{E}$ such that at a point $u_* \in \mathcal{E}$, the kernel $N(A_0)$ of the linearization A_0 is isomorphic to the tangent space of $\mathcal{E}$ at u_*, the eigenvalue 0 of A_0 is semi-simple, and the remaining spectral part of the linearization A_0 is stable. Then solutions starting nearby u_* exist globally and converge to some point on $\mathcal{E}$. This result is well known to specialists in the area of dynamical systems (where it is considered a folk theorem), but might be less familiar to people in the PDE community.

The situation described above occurs frequently in applications. We call it the *generalized principle of linearized stability*, and the equilibrium u_* is then termed *normally stable*.

A typical example for this situation to occur is the case where the equations under consideration involve symmetries, i.e. are invariant under the action of a Lie group $\mathcal{G}$. If then u_* is an equilibrium, the manifold $\mathcal{E}$ includes the action of $\mathcal{G}$ on u_* and the manifold $\mathcal{G}u_*$ is a subset of $\mathcal{E}$.

A standard method to handle situations as described above is to refer to *center manifold theory*. In fact in that situation the center manifold of the problem in question will be unique, and it coincides with $\mathcal{E}$ near u_*. Thus the so-called *shadowing lemma* in center manifold theory implies the result. Center manifolds are well-studied objects in the theory of nonlinear evolution equations. For the parabolic case we refer to the monographs [18,21], and to the publications [6,7,10,19,20,22,28,29].

However, the theory of center manifolds is a technically difficult matter. It usually involves higher regularity of the involved nonlinearities—in particular concerning the shadowing property. Therefore it seems desirable to have a simpler, direct approach to the generalized principle of linearized stability which avoids the technicalities of center manifold theory.

The purpose of this paper is to present such an approach. It turns out that the effort is only slightly larger than that for the proof of the standard linearized stability result—which is simple. We emphasize that our approach requires only C^1-regularity for the nonlinearities. By several examples we will illustrate that our result is applicable to a variety of interesting problems in different areas of applied analysis. It is our belief that the approach devised in this manuscript will be fruitful for the stability analysis of equilibria for parabolic evolution equations that involve symmetries in the way described above.

Here we would also like to mention the work in [9], where the action of a Lie group has been used for the stability analysis of equilibrium solutions. However, the approach given here is considerably more general and flexible.

In Section 2 we formulate and prove our main result for abstract autonomous quasilinear parabolic problems. Theorem 2.1 implies, for instance, the main result in [15] on convergence of solutions for the Mullins–Sekerka problem. We also show by means of examples that the conditions of Theorem 2.1 are necessary in order to have convergence to a single equilibrium.

In Section 3, we consider quasilinear parabolic systems with nonlinear boundary conditions and we show that our techniques can also be applied to this situation. Sections 4 and 5 illustrate the scope of our main result, as we show convergence towards equilibria for the Mullins–Sekerka model, and stability of travelling waves for a quasilinear parabolic equation.

In Section 6 we consider the so-called *normally hyperbolic* case, where the remaining part of the spectrum of A_0 also contains an unstable part away from the imaginary axis. In this situation, one cannot expect convergence of all solutions starting near u_*, but only for those initial values which are on the stable manifold.

To cover the quasilinear case our approach makes use of maximal L_p-regularity in an essential way. As general references for this theory we refer to the recent publications [11,12], to the survey article [24], and also to [2–4,8,21].

In a forthcoming paper these results are extended to the case where the boundary conditions are of relaxation type, i.e. are coupled with an evolution equation on the boundary, as in [13]. Problems of the last kind are important e.g. for the Stefan problem with surface tension, see [14,26], and for the two-phase Navier–Stokes problem with a free boundary.

Finally, we should like to point out that the generalized principle of linearized stability described in the current paper can also be adapted and applied to fully nonlinear parabolic equations, see [27].