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Exact boundary controllability of the nonlinear Schrödinger equation

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ABSTRACT

This paper studies the exact boundary controllability of the semi-linear Schrödinger equation posed on a bounded domain $\Omega \subset \mathbb{R}^n$ with either the Dirichlet boundary conditions or the Neumann boundary conditions. It is shown that if

$$s > \frac{n}{2},$$

or

$$0 \leqslant s < \frac{n}{2} \quad \text{with } 1 \leqslant n < 2 + 2s,$$

or

$$s = 0, 1 \quad \text{with } n = 2,$$

then the systems are locally exactly controllable in the classical Sobolev space $H^s(\Omega)$ around any smooth solution of the cubic Schrödinger equation.

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1. Introduction

In this paper we will study the boundary control system described by the nonlinear Schrödinger equation

$$iu_t + \Delta u + \lambda |u|^2 u = 0 \tag{1.1}$$

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posed on a bounded domain $\Omega \subset \mathbb{R}^n$ with either the Dirichlet boundary conditions

$$u(x,t) = h(x,t), \quad x \in \partial\Omega, \tag{1.2}$$

or the Neumann boundary conditions

$$\frac{\partial u(x,t)}{\partial \nu} = h(x,t), \quad x \in \partial\Omega, \tag{1.3}$$

where $u = u(x,t)$ is a complex-valued function of two real variables $x \in \Omega$ and $t \in \mathbb{R}$, the subscripts denote the corresponding partial derivatives, the parameter λ is a nonzero complex constant while the boundary value function $h(x,t)$ is considered as a control input.

We are mainly concerned with the following exact control problem for systems (1.1)–(1.2) and (1.1)–(1.3).

Let $s \geqslant 0$ and $T > 0$ be given. For any $u_0 \in H^s(\Omega)$ and $u_1 \in H^s(\Omega)$, can one find an appropriate boundary control h such that the system (1.1)–(1.2) *or system* (1.1)–(1.3) *admits a solution $u \in C([0,T]; H^s(\Omega))$ satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x)?$$

Control and stabilization problems for the Schrödinger equation have received a lot of attentions in the past decade.[1] While significant progresses have been made for the linear Schrödinger equation on its controllability and stabilizability properties (cf. e.g. [1,3,10,12,13,15–21,23] and the references therein), there are only a few results for the nonlinear Schrödinger equation, a situation in sharp contrast with other nonlinear dispersive equations, e.g. the Korteweg–de Vries equation (see [7,22,24, 25,29,31]), or the Ginzburg–Landau equation (see [6,26]). Recently, Illner, Lange and Teismann [8,9] considered internal controllability of the nonlinear Schrödinger equation posed on a finite interval $(-\pi,\pi)$:

$$iv_t + v_{xx} + \lambda|v|^2 v = f(x,t), \quad x \in (-\pi,\pi), \tag{1.4}$$

with the periodic boundary conditions

$$v(-\pi,t) = v(\pi,t), \qquad v_x(-\pi,t) = v_x(\pi,t), \tag{1.5}$$

where the forcing function $f = f(x,t)$, supported in a subinterval of $(-\pi,\pi)$, is considered as a control input. They showed that the system (1.4)–(1.5) is locally exactly controllable in the space $H^1_p(-\pi,\pi) := \{v \in H^1(-\pi,\pi)\colon\ v(-\pi) = v(\pi)\}$. Later, Lange and Teismann [11] considered internal control of the nonlinear Schrödinger equation (1.4) posed on a finite interval with the homogeneous Dirichlet boundary conditions

$$v(-\pi,t) = 0, \qquad v(\pi,t) = 0. \tag{1.6}$$

They showed that the system (1.4)–(1.6) is locally exactly controllable in the space $H^1_0(0,\pi)$ around a special ground state of the system.

Dehman, Gérard and Lebeau [5] studied internal control and stabilization of a class of defocusing nonlinear Schrödinger equations posed on a two-dimensional compact Riemannian manifold M without boundary:

$$iw_t + \Delta w - |w|^2 w = f(x,t), \quad x \in M. \tag{1.7}$$

[1] The readers are referred to Zuazua [32] for an excellent review on recent progresses of this subject up to 2003.