Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Differential Equations 231 (2006) 195–211
Journal of Differential Equations
www.elsevier.com/locate/jde

# Classification of blow-up with nonlinear diffusion and localized reaction

Raúl Ferreira [a], Arturo de Pablo [b,*], Juan Luis Vazquez [c]

[a] *Departamento Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[b] *Departamento Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain*
[c] *Departamento Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain*

Received 23 January 2006

Available online 13 June 2006

**Abstract**

We study the behaviour of nonnegative solutions of the reaction–diffusion equation

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

The model contains a porous medium diffusion term with exponent $m > 1$, and a localized reaction $a(x)u^p$ where $p > 0$ and $a(x) \geqslant 0$ is a compactly supported symmetric function. We investigate the existence and behaviour of the solutions of this problem in dependence of the exponents $m$ and $p$. We prove that the critical exponent for global existence is $p_0 = (m + 1)/2$, while the Fujita exponent is $p_c = m + 1$: if $0 < p \leqslant p_0$ every solution is global in time, if $p_0 < p \leqslant p_c$ all solutions blow up and if $p > p_c$ both global in time solutions and blowing up solutions exist. In the case of blow-up, we find the blow-up rates, the blow-up sets and the blow-up profiles; we also show that reaction happens as in the case of reaction extended to the whole line if $p > m$, while it concentrates to a point in the form of a nonlinear flux if $p < m$. If $p = m$ the asymptotic behaviour is given by a self-similar solution of the original problem.


*Keywords:* Blow-up; Porous medium equation; Asymptotic behaviour; Localized reaction; Nonlinear boundary conditions

* Corresponding author.
*E-mail addresses:* raul_ferreira@mat.ucm.es (R. Ferreira), arturop@math.uc3m.es (A. de Pablo), juanluis.vazquez@uam.es (J.L. Vazquez).

0022-0396/$ – see front matter 
doi:10.1016/j.jde.2006.04.017

## 1. Introduction and main results

This paper is motivated by the wish to understand the blow-up properties of reaction–diffusion equations which combine a localized reaction term with nonlinear diffusion. In order to fix ideas, the present study concentrates the nonlinear reaction–diffusion equation

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Nonnegative solutions $u \geqslant 0$ are considered. We take exponents $m > 1$ and $p > 0$ and the coefficient $a(x) \geqslant 0$ is a compactly supported function; this means that the reaction term acts only *locally*, and this is the main difference with existing studies of blow-up for similar reaction–diffusion equations. Thus, the problem may be used to describe a chemical reaction–diffusion process in which, due to the effect of the catalyst, the reaction takes place only at some local sites [2]. We remark that the name localized has received also some other interpretation in the literature of blow-up: a reaction depending only on the value of the unknown $u$ in some local set, for instance a point $R(u(x,t)) = u^p(x_0,t)$, but acting throughout the whole domain of interest, see the survey [22]. In our case, we have $R(u(x,t)) = 0$ outside the support of $a(x)$. As to the corresponding $n$-dimensional model $u_t = \Delta u^m + a(x)u^p$, there exist several interesting possible choices of the localized reaction, see [3]. Some of the results of this paper extend to those situations, this is the subject of a future work. Finally, the initial value $u_0$ is assumed to be continuous and nonnegative. More precise assumptions are made below.

Since $m > 1$, we have slow diffusion: if, for instance, $u_0$ has compact support then the function $u$ is, in general, a solution only in a weak sense, i.e., $u$ and $(u^m)_x$ are absolutely continuous functions and the equation is understood in the weak sense; $u$ is $C^\infty$ in its positivity set but not globally. Local in time existence, as well as a comparison principle, can easily be obtained, but the solution may only exist for $t \in [0,T)$ and become unbounded as $t \to T$ for some $T < \infty$. In other words, the solution may *blow up* in finite time, and this is our main concern.

Let us examine what is known in some standard case before presenting our results. It is well known that blow-up happens for the problem with *homogeneous reaction*, i.e., $a \equiv 1$:

$$\begin{cases} u_t = (u^m)_{xx} + u^p, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

All the solutions to this problem blow up if $1 < p \leqslant m+2$, while for $p > m+2$ they blow up provided that the initial data $u_0$ are large enough. In this case, the numbers $p_0 = 1$ and $p_c = m+2$ are called the *global existence exponent* and the *Fujita exponent*, respectively. This analysis can be extended to Eq. (1) with $a(x) \geqslant \delta > 0$. Our investigation will show that the exponents of the problem with localized reaction are not the same.

On the other hand, there is a close connection of problem (1) with the problem of diffusion with nonlinear boundary flux conditions. Thus, if we take a sequence of reaction coefficients converging to a Dirac delta at the origin (i.e., if $a_n(x) \to \delta_0(x)$), it is clear from the weak formulation of the problem, at least formally, that the corresponding solutions $u_n$ should converge to a solution of the problem

$$\begin{cases} u_t = (u^m)_{xx}, & (x,t) \in (0,\infty) \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t), & t \in \times(0,T), \\ u(x,0) = u_0(x), & x \in (0,\infty). \end{cases} \tag{3}$$