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# Uniqueness and blow-up rate of large solutions for elliptic equation $-\Delta u = \lambda u - b(x)h(u)$

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ABSTRACT

In this paper, we establish the blow-up rate of the large positive solution of the singular boundary value problem

$$\begin{cases} -\Delta u = \lambda u - b(x)h(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$. The weight function $b(x)$ is a non-negative continuous function in the domain. $h(u)$ is locally Lipschitz continuous and $h(u)/u$ is increasing on $(0,\infty)$ and $h(u) \sim Hu^p$ for sufficiently large $u$ with $H > 0$ and $p > 1$. Naturally, the blow-up rate of the problem equals its blow-up rate for the very special, but important, case when $h(u) = Hu^p$. We distinguish two cases: (I) $\Omega$ is a ball domain and $b$ is a radially symmetric function on the domain in Theorem 1.1; (II) $\Omega$ is a smooth bounded domain and $b$ satisfies some local condition on each boundary normal section assumed in Theorem 1.2. The blow-up rate is explicitly determined by functions $b$ and $h$. In case (I), the singular boundary value problem has a unique solution $u$ satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{KH^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} = 1,$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, $b^*(r)$ and $K$ are defined by

$$b^*(r) = \int_r^R \int_s^R b(t)\,dt\,ds,$$

$$K = \left[\beta\big((\beta+1)C_0 - 1\big)\right]^{\frac{1}{p-1}}, \quad \beta := \frac{1}{p-1}.$$

In case (II), the blow-up rates of the solutions to the boundary value problem are established and the uniqueness is proved.



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## 1. Introduction and main results

This paper continues the studies of the semilinear elliptic problems with singular boundary value condition in the following form:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The boundary condition in (1.1) is understood as $u(x) \to +\infty$ when $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0^+$. The non-negative solutions of (1.1) are called *large* (or *blow-up*) *solutions*.

Singular boundary value problem (1.1) arises naturally from a number of different areas and has a long history. Considerable amounts of study have been attracted by such problems (see, e.g., [2–5,11, 14,18,26–28] and references therein). In 1916 Bieberbach [3] studied the large solutions for the particular case $f(x,u) = -\exp(u)$ and $N = 2$. He showed that there exists a unique solution of (1.1) such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Problems of this type arise in Riemannian geometry: if a Riemannian metric of the form $|ds|^2 = \exp(2u(x))|dx|^2$ has constant Gaussian curvature $-c^2$, then $-\Delta u = -c^2 \exp(2u)$. Motivated by a problem in mathematical physics, Rademacher [31] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$. In 1990s, Bandle and Essèn [2] and Lazer and McKenna [18] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniformal external sphere condition and for nonlinearities $f(x,u) = b(x)\exp(u)$, where $b$ is continuous and strictly positive on $\bar{\Omega}$. It is shown that the problem exhibits a unique solution in a smooth domain together with an estimate of the form $u = \log d^{-2} + o(d)$ in [18] (where $b(x) \geqslant b_0 > 0$ as $d \to 0$) and in [2] (where $b \equiv 1$).

For $f(x,u) = g(u)$, Lazer and McKenna [19] obtained an asymptotic result for solutions of the above problem under some assumptions on $g$. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N > 1$, which satisfies a uniform internal sphere condition and a uniform external sphere condition. Let $g$ be a $C^1$-function which is either defined and positive on $(-\infty, \infty)$ or is defined on a ray $[a, \infty)$ with $g(a) = 0$ and $g(s) > 0$ for $s > a$. They further assume that $g'(s) \geqslant 0$ for $s$ in the domain of $g$, and that there exists $a_1$ such that $g'(s)$ is non-decreasing for $s \geqslant a_1$. They proved that if $\lim_{s\to\infty} \frac{g'(s)}{\sqrt{G(s)}} = \infty$ where $G'(s) = g(s)$, $G(s) > 0$, then the problem has a unique solution $u(x)$ and moreover,

$$u(x) - Z\big(d(x)\big) \to 0 \quad \text{as } d(x) \to 0,$$

where $Z$ is a solution on an interval $(0,b)$, $b > 0$, of the equation $Z''(r) = g(Z(r))$ and such that $Z(r) \to \infty$ as $r \to 0^+$.

We are interested in large solutions of (1.1) when $f(x,u) = \lambda(x)u - b(x)h(u)$, i.e.

$$-\Delta u = \lambda(x)u - b(x)h(u) \quad \text{in } \Omega \tag{1.2}$$

subject to the singular boundary condition

$$u = +\infty \quad \text{on } \partial\Omega, \tag{1.3}$$

where $h \in C[0,\infty)$ is locally Lipschitz, $\lambda \in L^\infty$ and $b$ is a continuous function in $\bar{\Omega}$ with positive value in $\Omega$ and with non-negative value on $\partial\Omega$.

If $b > 0$ in $\bar{\Omega}$ and $h(u) = u^p$ $(p > 1)$, then (1.2) is known as the logistic equation. This equation is a basic population model (see, e.g., [11,21,24] and the references therein). Generally speaking, the existence problem is relatively well understood but the uniqueness problem is only partially understood. Assume that $h \geqslant 0$ is non-negative locally Lipschitz continuous and $h(u)/u$ is increasing on $(0,\infty)$. Then, necessarily $h(0) = 0$, and by the strong maximum principle, any non-negative classical solution