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Evolving a convex closed curve to another one via a length-preserving linear flow

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ABSTRACT

Motivated by a recent curvature flow introduced by Professor S.-T. Yau [S.-T. Yau, Private communication on his "Curvature Difference Flow", 2007], we use a simple curvature flow to evolve a convex closed curve to another one (under the assumption that both curves have the same length). We show that, under the evolution, the length is preserved and if the curvature is bounded above during the evolution, then an initial convex closed curve can be evolved to another given one.

1. Introduction

Let γ_1, γ_2 be two given smooth curves in $\mathbb{R}^2$ (by smooth, we mean C^∞). An interesting problem proposed by Professor S.-T. Yau [25] in plane curves evolution is to evolve γ_1 to converge to γ_2 (can be up to an isometry) eventually, either in finite time or in infinite time, using a parabolic curvature flow method. The evolution of plane curves via curvature flow method has been studied extensively in the papers, to name just a few, by Andrews [1], Angenent [3], Abresch and Langer [2], Chow and Tsai [5,7,21], Nien and Tsai [17], Gage [11,12], Gage and Hamilton [13], Grayson [10], Huisken [14], Jiang and Pan [15], Pan and Yang [18], Urbas [23], Yagisita [24], or in the book by Chou and Zhu [9], etc. However, in all of these literature items the settings are more or less the same, i.e., we first start with a given initial curve γ_0 in $\mathbb{R}^2$, and then evolve it along its normal vector direction with certain type of speed depending on the curvature (or depending on the non-local quantities length and enclosed area) of the evolving curve γ_t, and finally study its asymptotic convergence behavior.

On the other hand, the plane curves evolution problem suggested by Yau [25], which we call "Yau's Curvature Difference Flow (YCDF)", has different setting: First, we are given two smooth embedded

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curves γ_1, γ_2 in $\mathbb{R}^2$ parametrized by smooth embeddings $X_1(\varphi): I \to \mathbb{R}^2$ and $X_2(\varphi): I \to \mathbb{R}^2$, where I is either an interval in $\mathbb{R}$ or S^1. Then we consider a curvature flow of the form (γ_2 plays the role of a steady curve here)

$$\begin{cases} \dfrac{\partial X}{\partial t}(\varphi, t) = \big[k_1(\varphi, t) - k_2(\varphi)\big]\mathbf{N}_1(\varphi, t), \\ X(\varphi, 0) = X_1(\varphi), \quad \varphi \in I \end{cases} \tag{1}$$

where $k_1(\varphi, t)$ is the curvature of $\gamma_{1,t}$ (the evolution of γ_1 at time t, parametrized by $X(\varphi, t)$, $\gamma_{1,0} = \gamma_1$) at φ; $\mathbf{N}_1(\varphi, t)$ is the normal vector of $\gamma_{1,t}$ at φ; and $k_2(\varphi)$ is the curvature of γ_2 at φ.

The significance of YCDF (1) is that the flow stops when $\gamma_{1,t}$ converges to γ_2 (of course it is also likely that $\gamma_{1,t}$ may not evolve to γ_2 at all). As far as we know, this type of flow (i.e., YCDF) has not appeared elsewhere before. It is attractive and can certainly lead to a lot of interesting questions.

Motivated by YCDF and Urbas [23], we consider a related special evolution problem as follows. It is valid only in the convex setting. First we assume γ_1 and γ_2 are two given *convex*[2] closed curves in $\mathbb{R}^2$. Then we consider a flow of the form

$$(*) \quad \begin{cases} \dfrac{\partial X}{\partial t}(\varphi, t) = \left(\dfrac{1}{k_1(\varphi, t)} - \dfrac{1}{k_2(\varphi, t)}\right)\mathbf{N}_{1,out}(\varphi, t), \\ X(\varphi, 0) = X_1(\varphi), \quad \varphi \in S^1 \end{cases}$$

where $X_1(\varphi)$ is the parametrization of γ_1; $k_1(\varphi, t)$ is the curvature of $\gamma_{1,t}$ at φ; $\mathbf{N}_{1,out}(\varphi, t)$ is the outward normal of $\gamma_{1,t}$ at φ; and $k_2(\varphi, t)$ is the curvature of the stationary convex curve γ_2 at the unique point where its outward normal is $\mathbf{N}_{1,out}(\varphi, t)$.

Remark 1. Note the difference: in YCDF we are comparing curvatures at the same parameter φ, while in $(*)$ we are comparing curvatures at the same outward normal. That is why we have $1/k_2(\varphi, t)$ in $(*)$ instead of $1/k_2(\varphi)$ (note that the parameter φ in $(*)$ is not necessarily normal angle of γ_1).

Since both curves γ_1 and γ_2 are convex, one can use the outward normal angle θ to parametrize them simultaneously. Moreover, we can use the support function to reformulate the problem, i.e., the flow $(*)$ is equivalent to a scalar parabolic PDE for the support function on S^1. See [1,22], or [23] for details.

We first review some facts about support functions of convex plane curves. The support function U of a convex closed curve γ_0, expressed in terms of its outward normal angle $\theta \in S^1$, is defined as

$$U(\theta) = \big\langle P(\theta), (\cos\theta, \sin\theta)\big\rangle, \quad \theta \in S^1 \tag{2}$$

where $P(\theta)$ is the position vector of the unique point $p \in \gamma_0$ whose outward normal angle is θ. By (2), $U(\theta)$ is the normal component of the position vectors of γ_0, and we have

$$P(\theta) = U(\theta)(\cos\theta, \sin\theta) + U_\theta(\theta)(-\sin\theta, \cos\theta), \quad \theta \in S^1. \tag{3}$$

By (3) the average position vectors of $P(\theta)$ over S^1 can be expressed as (tangential part and normal part have the same contribution)

$$\frac{1}{2\pi}\int_{S^1} P(\theta)\, d\theta = \frac{1}{\pi}\int_{S^1} U(\theta)(\cos\theta, \sin\theta)\, d\theta. \tag{4}$$

[2] In this paper "convex" always means "strictly convex", i.e., curvature is positive everywhere (no inflection points). A convex closed curve may have self-intersections, but in this paper we are confined to the *embedded* case, i.e., it is a simple closed curve.