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Extremal values of smallest eigenvalues of Hill's operators with potentials in L^1 balls ☆

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ABSTRACT

Given a 1-periodic real potential $q \in L^1(\mathbb{R}/\mathbb{Z})$. We use $\lambda_0(q)$ to denote the smallest 1-periodic eigenvalue of the Hill's equation $x'' + (\lambda + q(t))x = 0$. Let $B_1[r]$ be the ball centered at 0 of radius r in the L^1 space $L^1(\mathbb{R}/\mathbb{Z})$. It is trivial that $\sup\{\lambda_0(q)\colon q \in B_1[r]\} = r$ for all $r \geqslant 0$. Based on continuity of $\lambda_0(q)$ in q with the weak topology and continuous differentiability of $\lambda_0(q)$ in q with the L^1 norm $\|\cdot\|_1$, we will apply scaling technique, variational approach to extremal values in L^p balls, singular integrals and the limiting approach as $p \downarrow 1$ to obtain (i) $\lambda_0(q)$ is bounded for q in any bounded set of $(L^1(\mathbb{R}/\mathbb{Z}), \|\cdot\|_1)$, and (ii) the minimal value

$$\mathbf{L}_1(r) := \inf\{\lambda_0(q)\colon q \in L^1(\mathbb{R}/\mathbb{Z}),\ \|q\|_1 \leqslant r\}$$
$$= \inf\{\lambda_0(q)\colon q \in L^1(\mathbb{R}/\mathbb{Z}),\ \|q\|_1 = r\}$$

is simply $\mathbf{Z}_0^{-1}(r)$, where $\mathbf{Z}_0(x) := 2\sqrt{-x}\tanh(\sqrt{-x}/2)$, $x \leqslant 0$. The extremal values of the smallest Neumann eigenvalues for potentials in L^1 balls are also found explicitly.

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1. Introduction

Eigenvalues and their estimates are important in many problems in mathematics and applied sciences. In this paper, we will use continuity of eigenvalues in weak topologies, some topological facts on L^p spaces, variational method, dynamical systems and singular integrals to give some deep results on the smallest periodic eigenvalues of Hill's operators.

Let q be a 1-periodic (real) potential from the Lebesgue space $\mathcal{L}^p := L^p(\mathbb{S}_1)$, where $\mathbb{S}_1 = \mathbb{R}/\mathbb{Z}$ and $1 \leqslant p \leqslant \infty$. The eigenvalue problem

$$x'' + \big(\lambda + q(t)\big)x = 0 \tag{1.1}$$

has a double-sequence of eigenvalues

$$\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \leqslant \overline{\lambda}_1(q) < \cdots < \underline{\lambda}_m(q) \leqslant \overline{\lambda}_m(q) < \cdots,$$

where $\underline{\lambda}_m(q)$, $\overline{\lambda}_m(q)$ are 1-periodic eigenvalues of (1.1) for m even, and $\underline{\lambda}_m(q)$, $\overline{\lambda}_m(q)$ are 1-anti-periodic eigenvalues of (1.1) for m odd. See [15,24], or [25] for a rotation number approach.

As a functional of potentials $q \in \mathcal{L}^p$, each of these eigenvalues is continuous in the usual L^p topology $\|q\|_p := \|q\|_{L^p(\mathbb{S}_1)}$. Moreover, since $\overline{\lambda}_0(q)$ is simple, $q \in (\mathcal{L}^p, \|\cdot\|_p) \to \overline{\lambda}_0(q)$ is actually continuously differentiable [13]. A recent result by the author shows that eigenvalues have very strong continuous dependence on potentials.

Theorem 1.1. *(See Zhang [27].) Let $1 \leqslant p \leqslant \infty$ and $m \geqslant 0$. The functionals*

$$\big(\mathcal{L}^p, w_p\big) \to \mathbb{R}, \qquad q \to \underline{\lambda}_m(q), \qquad q \to \overline{\lambda}_m(q)$$

are continuous, where w_p indicates the topology of weak convergence in the space $(\mathcal{L}^p, \|\cdot\|_p)$ for $1 \leqslant p < \infty$ and w_∞ is the topology of weak convergence in the space $(\mathcal{L}^\infty, \|\cdot\|_\infty)$. Here $\underline{\lambda}_0(q)$ is void.*

For case $p = \infty$, see also [17]. For case $2 \leqslant p \leqslant \infty$, see also [18]. For some continuity results of solutions in weak topologies, see [11,20].

Such a strong continuity of eigenvalues has some important implications. In case $1 < p \leqslant \infty$, it is well known [23] that any bounded subset of $(\mathcal{L}^p, \|\cdot\|_p)$ is sequentially relatively compact in $(\mathcal{L}^p, w_p)$. By Theorem 1.1, both $\underline{\lambda}_m(q)$ and $\overline{\lambda}_m(q)$ are bounded for q in any bounded subset of $(\mathcal{L}^p, \|\cdot\|_p)$. Since bounded subsets of $(\mathcal{L}^1, \|\cdot\|_1)$ may lack compactness even in w_1, the boundedness of $\underline{\lambda}_m(q)$ and $\overline{\lambda}_m(q)$ for q in bounded subsets of $(\mathcal{L}^1, \|\cdot\|_1)$ cannot be deduced from Theorem 1.1 in a direct way. In this paper, we will completely solve this for the smallest periodic eigenvalues $\overline{\lambda}_0(q)$.

For $1 \leqslant p \leqslant \infty$ and $r \geqslant 0$, let

$$B_p[r] := \big\{q \in \mathcal{L}^p\colon\ \|q\|_p \leqslant r\big\}$$

be the ball of the L^p space. Since we are mainly concerned with the smallest periodic eigenvalues in this paper, for simplicity, we write $\overline{\lambda}_0(q)$ as $\lambda_0(q)$ for $q \in \mathcal{L}^p$. Let us define the following extremal values

$$\mathbf{L}_p(r) := \inf_{q \in B_p[r]} \lambda_0(q), \qquad \mathbf{M}_p(r) := \sup_{q \in B_p[r]} \lambda_0(q). \tag{1.2}$$

For the maximal values $\mathbf{M}_p(r)$, let us recall a trivial upper bound of $\lambda_0(q)$

$$\lambda_0(q) \leqslant -\int_0^1 q(t)\,\mathrm{d}t, \quad q \in \mathcal{L}^p. \tag{1.3}$$