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Stability of parabolic problems with nonlinear Wentzell boundary conditions

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ABSTRACT

Of concern is the nonlinear uniformly parabolic problem

$$u_t = \operatorname{div}(\mathcal{A}\nabla u), \qquad u(0,x) = f(x),$$

$$u_t + \beta\partial_\nu^{\mathcal{A}} u + \gamma(x,u) - q\beta\Delta_{\mathrm{LB}} u = 0,$$

for $x \in \Omega \subset \mathbb{R}^N$ and $t \geqslant 0$; the last equation holds on the boundary $\partial\Omega$. Here $\mathcal{A} = \{a_{ij}(x)\}_{ij}$ is a real, hermitian, uniformly positive definite $N \times N$ matrix; $\beta \in C(\partial\Omega)$ with $\beta > 0$; $\gamma : \partial\Omega \times \mathbb{R} \to \mathbb{R}$; $q \in [0,\infty)$, Δ_{LB} is the Laplace–Beltrami operator on the boundary, and $\partial_\nu^{\mathcal{A}} u$ is the conormal derivative of u with respect to A: and everything is sufficiently regular. The solution of this wellposed problem depends continuously on the ingredients of the problem, namely, $\mathcal{A}, \beta, \gamma, q, f$. This is shown using semigroup methods in [G.M. Coclite, A. Favini, G.R. Goldstein, J.A. Goldstein, S. Romanelli, Continuous dependence on the boundary parameters for the Wentzell Laplacian, Semigroup Forum 77 (1) (2008) 101–108]. Here we prove explicit stability estimates of the solution u with respect to the coefficients $\mathcal{A}$, β, γ, q, and the initial condition f. Moreover we cover the singular case of a problem with $q = 0$ which is approximated by problems with positive q.

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1. Introduction

This paper deals with real solutions of the following parabolic problem with nonlinear Wentzell (or dynamics) boundary conditions

$$\begin{cases} u_t = \operatorname{div}(\mathcal{A}\nabla u), & \text{in } (0,\infty)\times\Omega, \\ u(0,\cdot) = f, & \text{in } \Omega, \\ u_t + \beta\partial_\nu^{\mathcal{A}} u + \gamma(x,u) - q\beta\Delta_{\mathrm{LB}} u = 0, & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{1.1}$$

We extend the stability results obtained in [5], where we considered the linear problem corresponding to the choice $\gamma(x,u) = \gamma(x)u$.

Such an initial–boundary value problem can model a diffusion process, for example, the heat equation with a heat source on the boundary. When the heat source on the boundary depends nonlinearly on the heat flow across and the temperature on the boundary and the heat can transfer along the boundary, we obtain a boundary condition of the form in (1.1) (see [16] for a derivation of such boundary conditions). Many authors have considered parabolic problems with dynamic (and the related general Wentzell) boundary conditions (cf. [9–12,14,22–25]).

In this paper we assume

(i) $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded open set with C^2 boundary;

(ii) $\mathcal{A} = \{a_{ij}\}_{ij} \in C^1(\overline{\Omega}; \mathbb{R}^{N\times N})$ is symmetric and uniformly elliptic, in particular there exist two constants $\alpha_1 \geqslant \alpha_0 > 0$ such that

$$\alpha_0|\xi|^2 \leqslant \langle \mathcal{A}(x)\xi, \xi\rangle \leqslant \alpha_1|\xi|^2,$$

for each $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$;

(iii) $\beta \in C(\partial\Omega)$ and $\beta_1 \geqslant \beta(x) \geqslant \beta_0 > 0$, for some constants β_1, β_0, and every $x \in \partial\Omega$;

(iv) $0 \leqslant q < \infty$ is a given constant;

(v) Δ_{LB} is the Laplace–Beltrami operator on $\partial\Omega$;

(vi) $\partial_\nu^{\mathcal{A}}$ is the conormal derivative operator on the boundary, namely

$$\partial_\nu^{\mathcal{A}} u = \langle \mathcal{A}\nabla u, \nu\rangle = \sum_{ij} a_{ij}\partial_{x_i} u \nu_j,$$

where ν is the unit outer normal at $x \in \partial\Omega$;

(vii) $f \in H^1(\Omega)$; concerning the trace of f on $\partial\Omega$, when $q > 0$ we require that $\nabla_\tau f \in L^2(\partial\Omega)$, where ∇_τ is the tangential gradient;

(viii) $\gamma : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, $\gamma(\cdot,0) = 0$, for each $x \in \partial\Omega$ the function $\gamma(x,\cdot)$ is nondecreasing, and

$$\left|\gamma(x,u)\right| \leqslant \Gamma\left(|u|^p + 1\right), \tag{1.2}$$

for some positive constants Γ, p such that

$$p \begin{cases} \leqslant \frac{N}{N-2} & \text{if } N \geqslant 3, \\ < \infty & \text{otherwise.} \end{cases} \tag{1.3}$$

All of the above coefficients are assumed to be real-valued.

As a consequence of (viii) we have that

$$u \geqslant 0 \quad \text{implies} \quad \gamma(x,u) \geqslant 0, \qquad u \leqslant 0 \quad \text{implies} \quad \gamma(x,u) \leqslant 0, \tag{1.4}$$

and in particular