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Instability of equilibrium points of some Lagrangian systems

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Abstract

In this work we show that, if $\mathcal{L}$ is a natural Lagrangian system such that the k-jet of the potential energy ensures it does not have a minimum at the equilibrium and such that its Hessian has rank at least $n-2$, then there is an asymptotic trajectory to the associated equilibrium point and so the equilibrium is unstable. This applies, in particular, to analytic potentials with a saddle point and a Hessian with at most 2 null eigenvalues.

The result is proven for Lagrangians in a specific form, and we show that the class of Lagrangians we are interested can be taken into this specific form by a subtle change of spatial coordinates. We also consider the extension of this results to systems subjected to gyroscopic forces.

Keywords: Liapunov stability; Lagrange–Dirichlet theorem; Lagrangian systems

1. The problem

Consider the study of the Liapunov instability of equilibrium points of conservative Lagrangian systems in $\mathbb{R}^{2n}$, with Lagrangians $\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - \pi(q)$, where π is the potential energy and T the kinetic energy.

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Lagrange's equations for these systems are

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}-\frac{\partial\mathcal{L}}{\partial q}=0 \tag{1}$$

and its equilibrium points are of the form $(q_0,0)$ where $\frac{\partial\pi}{\partial q}(q_0)=0$.

In this context, the classic Lagrange–Dirichlet theorem assures that, if q_0 is a strict local minimum of π, then $(q_0,0)$ is stable in the sense of Liapunov. The ongoing study of conditions for the instability of this equilibrium when q_0 is not a strict local minimum of the potential started in 1904, when Painleve exhibit a counter-example to the full reciprocal of the aforementioned theorem.

In the context of analytic Lagrangians, in [9] it is shown that if the potential energy does not have a minimum in q_0, then $(q_0,0)$ is unstable.

The case when π is not analytic has many more particularities. For instance, in [6], the author considered the kinetic energy $T(q_1,q_2,\dot{q}_1,\dot{q}_2)=\frac{\dot{q}_1^2+\dot{q}_2^2}{2}$ and the potential energy $\pi(q_1,q_2)=e^{-\frac{1}{q_1^2}}\cos\frac{1}{q_1}-e^{-\frac{1}{q_2^2}}(\cos\frac{1}{q_2}+q_2^2)$. The associated system is such that the origin is an equilibrium point, $\pi(0)=0$, $\pi(q,q)<0$ if $q\neq 0$, and yet the equilibrium is stable.

A research project that has been leading to interesting results stems from a conjecture posed by Liapunov, which states that being the origin an equilibrium point for the Lagrangian system, if the k-jet of the potential energy shows that it does not have a minimum in the origin, then the origin is an unstable equilibrium point, Liapunov proved this conjecture for $k=2$ (see the next section for the definitions of k-jet and of it showing that the origin does not have a minimum). This conjecture, if true, would be the best result possible in the set of functions that have k-jet and it includes the analytical case, since in [2] it is proved that if π is analytic and does not have a minimum in the origin, then there is a positive integer k such that $j^k\pi$ shows this fact.

In the context of 2 degrees of freedom, the conjecture was completely proved in [5]. This was done using a Četaev like function to show the existence of an asymptotic trajectory to the origin.

In the general case of n degrees of freedom, we have only partial results in the direction of this conjecture. For example, in [8] and independently in [11], it is proved that when the jet that shows that the origin is not a minimum for the potential energy is homogeneous, then there is an asymptotic trajectory to $(q_0,0)$.

Extending this result, in [7] the following theorem is proved:

Theorem *(Maffei, Moauro and Negrini). Consider the Lagrangian system given by Eq.* (1) *in* $\mathbb{R}^{2(m+n)}$, *with* $q=(u,v)$, $\dot{q}=(\dot{u},\dot{v})$ *and* $\mathcal{L}=T-\pi$. *Assume there are an integer* $k\geqslant 3$ *and reals* $\omega_1,\dots,\omega_m$ *such that*:

1. $\pi(u,v)=\frac{1}{2}\langle u,l(u,v)u\rangle+\pi_{[k]}(u,v)+R(u,v)$, *where* $l(u,v)$ *is an* $m\times m$ *matrix such that* $l(0,0)=\operatorname{diag}(\omega_1^2,\dots,\omega_m^2)$, $\pi_{[k]}$ *homogeneous of degree* k, $R(u,v)=O(\|(u,v)\|^{k+1})$ *and* $\min\{\pi_{[k]}(0,v)\colon \|v\|=1\}=-1$.
2. $\mathcal{L}$ *is* $C^{k+3+\lfloor\frac{k-3}{2}\rfloor}$.

Then there is a trajectory $\phi(t)$ *such that* $(\phi(t),\dot{\phi}(t))\to(0,0)$ *as* $t\to-\infty$.

This result, in particular, proves the conjecture under the assumption that there is a split of $\mathbb{R}^{n+m}=\mathbb{R}^n\oplus\mathbb{R}^m$ where we have that the k-jet of π is $\pi_2(q_1,\dots,q_n)+\pi_k(q_{n+1},\dots,q_m)$, such