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Global solutions of singular parabolic equations arising from electrostatic MEMS

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Abstract

We study dynamic solutions of the singular parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda_* |x|^\alpha}{(1-u)^2}, & (x,t) \in B \times (0,\infty), \\ u(x,0) = u_0(x) \geqslant 0, & x \in B, \\ u(x,t) = 0, & x \in \partial B, \end{cases} \tag{P}$$

where $\alpha \geqslant 0$ and $\lambda_* > 0$ are two parameters, and B is the unit ball $\{x \in \mathbb{R}^N\colon\ |x| \leqslant 1\}$ with $N \geqslant 2$. Our interest is focussed on (P) with $\lambda_* := \frac{(2+\alpha)(3N+\alpha-4)}{9}$, for which (P) admits a singular stationary solution in the form $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$. This equation models dynamic deflection of a simple electrostatic Micro-Electro-Mechanical-System (MEMS) device. Under the assumption $u_0 \lneqq S(x)$, we address the existence, uniqueness, regularity, stability or instability, and asymptotic behavior of weak solutions for (P). Given $\alpha^{**} := \frac{4-6N+3\sqrt{6}(N-2)}{4}$, in particular we show that if either $N \geqslant 8$ and $\alpha > \alpha^{**}$ or $2 \leqslant N \leqslant 7$, then the minimal compact stationary solution u_{λ_*} of (P) is stable and while $S(x)$ is unstable. However, for $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, (P) has no compact minimal solution, and $S(x)$ is an attractor from below not from above. Furthermore, the refined asymptotic behavior of global solutions for (P) is also discussed for the case where $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, which is given by a certain matching of different asymptotic developments in the large outer region closer to the boundary and the thin inner region near the singularity.

Keywords: MEMS; Existence and uniqueness; Asymptotic behavior; Minimal solutions; Stability

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1. Introduction

The singular parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda f(x)}{(1-u)^2}, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

was recently proposed in [11,14], where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f(x)$ is a nonnegative function satisfying

$$f \in C^\alpha(\bar{\Omega}) \quad \text{for some } \alpha \in (0,1],\ 0 \leqslant f \leqslant 1, \quad \text{and}$$
$$f > 0 \quad \text{on a subset of } \Omega \text{ of positive measure.} \tag{1.2}$$

When $N = 1$ or 2, this equation models a simple electrostatic Micro-Electro-Mechanical-System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at $+1$. The dynamic solution $u(x,t)$ of (1.1) characterizes the dynamic deflection of the elastic membrane. When a voltage—represented here by λ—is applied to the surface of the membrane, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices. In an effort to achieve better MEMS designs, the material properties of the membrane can be technologically fabricated with a spatially varying dielectric permittivity profile $f(x)$ (see [11,14] and references therein for more detailed discussions on MEMS devices).