J. Differential Equations 245 (2008) 3095–3126

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Hyperbolic conservation laws with discontinuous fluxes and hydrodynamic limit for particle systems

Gui-Qiang Chen[a,*], Nadine Even[b], Christian Klingenberg[b]

[a] Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA
[b] Department of Mathematics, University of Würzburg, Am Hubland, D-97074 Würzburg, Germany

ARTICLE INFO

Article history:
Received 18 September 2007
Revised 23 May 2008
Available online 23 September 2008

MSC:
35L65
60K35
82C22

Keywords:
Hyperbolic conservation laws
Discontinuous flux functions
Measure-valued
Entropy solutions
Entropy conditions
Uniqueness
Hydrodynamic limits
Microscopic
Particle systems
Zero range process
Discontinuous speed-parameter
Compactness framework

ABSTRACT

We study the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F(x,\rho) = 0. \tag{0.1}$$

The main feature of such a conservation law is the discontinuity of the flux function in the space variable x. Kruzkov's approach for the L^1-contraction does not apply since it requires the Lipschitz continuity of the flux function in x; an additional jump wave may occur in the solution besides the classical waves; and entropy solutions even for the Riemann problem are not unique under the classical entropy conditions. On the other hand, it is known that, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed-parameter $\lambda(x)$ in the hydrodynamic limit formally lead to scalar hyperbolic conservation laws with discontinuous fluxes of the form

$$\partial_t \rho + \partial_x\big(\lambda(x)h(\rho)\big) = 0. \tag{0.2}$$

The natural question arises which entropy solution the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. This paper is a first step and provides an answer to this question for a family of discontinuous flux functions. In particular, we identify the entropy condition for (0.1) and proceed to show the well-posedness by combining our existence result with a uniqueness result of Audusse and Perthame (2005) for the family of flux functions; we establish a compactness framework for the

* Corresponding author.
E-mail addresses: gqchen@math.northwestern.edu (G.-Q. Chen), even@mathematik.uni-wuerzburg.de (N. Even), klingenberg@mathematik.uni-wuerzburg.de (C. Klingenberg).

doi:10.1016/j.jde.2008.07.036

hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (0.1), which is based on the notion and reduction of measure-valued entropy solutions; and we finally establish the hydrodynamic limit for a ZRP with discontinuous speed-parameter governed by an L^∞ entropy solution to (0.2).

1. Introduction

We are concerned with the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F\big(x, \rho(t,x)\big) = 0 \tag{1.1}$$

and with initial data

$$\rho|_{t=0} = \rho_0(x), \tag{1.2}$$

where $F(x,\rho)$ is continuous at all points of $(\mathbb{R}\setminus\mathcal{N})\times\mathbb{R}$ except on a set $\mathcal{N}\subset\mathbb{R}$ of measure zero.

The main feature of (1.1) is the discontinuity of the flux function in the space variable x. This feature causes new important difficulties in conservation laws. Kruzkov's approach in [18] for the L^1-contraction does not apply; an additional jump wave may occur in the solution besides the classical waves; and entropy solutions even for the Riemann problem of (1.1) are not unique under the classical entropy conditions.

Several different entropy conditions have been suggested in the literature (see [1,2,4,5,10,15,17,21] and the references therein). One type of entropy conditions involves a rule how the solution should behave at the jump wave induced by the discontinuity in the flux, that is, the solution is required to satisfy an additional condition on its traces at the discontinuous points of the flux function, for which the existence of traces of the solution is needed. An alternative entropy condition in [2,4] is an adapted entropy condition that uses steady-state solutions to replace the constant parameter in the Kruzkov entropy inequality. This is quite an attractive notion since it does not require the traces of the entropy solution, which allows the solution only in L^∞. In this paper, we establish the well-posedness in L^∞ for conservation laws with a certain class of flux functions (cf. conditions (H1)–(H2) and (H3) or (H3′) in Section 2 below) by providing an existence proof to supplement the uniqueness result in [2].

The entropy condition based on the traces of solutions at the jump waves has lead to the existence and uniqueness of the solutions for a wider class of flux functions than those satisfying (H1)–(H2) and (H3) or (H3′) in Section 2. The Cauchy problem (even the Riemann problem) may lead to different solutions depending on which choice of the conditions on the traces of solutions is made (for example, see [2]). If one restricts oneself to the flux functions satisfying (H1)–(H2) and (H3′) in Section 2 (in which $F(x,\cdot)$ in (1.1) is monotone) and to the entropy solutions in the class of functions of bounded variation, the two notions of entropy conditions addressed above will lead to the same solution. This is not the case for the flux functions satisfying (H1)–(H2) and (H3) in which $F(x,\cdot)$ is non-monotone.

On the other hand, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed-parameter $\lambda(x)$ in the hydrodynamic limit formally lead to scalar hyperbolic conservation laws with discontinuous flux of the form

$$\partial_t \rho + \partial_x\big(\lambda(x)h(\rho)\big) = 0 \tag{1.3}$$