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On the Cauchy problem for the Boltzmann equation in the whole space: Global existence and uniform stability in $L^2_\xi(H^N_x)$

Renjun Duan

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, PR China

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Abstract

Based on a refined energy method, in this paper we prove the global existence and uniform-in-time stability of solutions in the space $L^2_\xi(H^N_x)$ to the Cauchy problem for the Boltzmann equation around a global Maxwellian in the whole space $\mathbb{R}^3$. Compared with the solution space used by the spectral analysis and the classical energy method, the velocity weight functions or time derivatives need not be included in the norms of $L^2_\xi(H^N_x)$, which is realized by introducing some temporal interactive energy functionals to estimate the macroscopic dissipation rate. The key proof is carried out in terms of the macroscopic equations together with the local conservation laws. It is also found that the perturbed macroscopic variables actually satisfy the linearized compressible Navier–Stokes equations with remaining terms only related to the microscopic part.

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E-mail address: mathrjduan@hotmail.com.

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1. Introduction

The Boltzmann equation for the hard-sphere monatomic gas in the whole space $\mathbb{R}^3$ takes the form

$$\partial_t f + \xi \cdot \nabla_x f = Q(f, f). \tag{1.1}$$

Here, the unknown $f = f(t, x, \xi)$ is a non-negative function standing for the number density of gas particles which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t > 0$. Q is the bilinear collision operator defined by

$$Q(f, g) = \int_{\mathbb{R}^3 \times S^2} \big(f' g'_* - f g_*\big)\big|(\xi - \xi_*) \cdot \omega\big| \, d\omega \, d\xi_*,$$

$$f = f(t, x, \xi), \qquad f' = f(t, x, \xi'), \qquad g_* = g(t, x, \xi_*), \qquad g'_* = g\big(t, x, \xi'_*\big),$$

$$\xi' = \xi - \big[(\xi - \xi_*) \cdot \omega\big]\omega, \qquad \xi'_* = \xi_* + \big[(\xi - \xi_*) \cdot \omega\big]\omega, \quad \omega \in S^2.$$

We define the perturbation $u = u(t, x, \xi)$ by

$$f = \mathbf{M} + \sqrt{\mathbf{M}} u, \tag{1.2}$$

where the global Maxwellian

$$\mathbf{M} = \frac{1}{(2\pi)^{3/2}} \exp\big(-|\xi|^2/2\big)$$

is normalized to have zero bulk velocity and unit density and temperature. Then the equation for the perturbation u reads

$$\partial_t u + \xi \cdot \nabla_x u = \mathbf{L} u + \Gamma(u, u), \tag{1.3}$$

where

$$\mathbf{L} u = \frac{1}{\sqrt{\mathbf{M}}}\big[Q(\mathbf{M}, \sqrt{\mathbf{M}} u) + Q(\sqrt{\mathbf{M}} u, \mathbf{M})\big],$$

$$\Gamma(u, u) = \frac{1}{\sqrt{\mathbf{M}}} Q(\sqrt{\mathbf{M}} u, \sqrt{\mathbf{M}} u).$$