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Generalized solvability and optimization of a parabolic system with a discontinuous solution

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Abstract

We consider the following linear parabolic system in a domain with a thin low-permeable insertion ("imperfect interface"):

$$\begin{aligned}
&\frac{\partial u}{\partial t}+q(\xi)u+\nabla\cdot\vec{\omega}=f(t,\xi),\quad \vec{\omega}=-\mathbf{K}\nabla u,\quad (t,\xi)\in Q_1\cup Q_2\subset\mathbb{R}^n,\\
&u|_{t=0}=0,\qquad u|_{\xi\in\partial\Omega}=0,\qquad \mathbf{K}=\{k_{ij}(\xi)\}_{i,j=1}^n,\\
&\big[(\vec{\omega},\vec{n})_{\mathbb{R}^n}\big]=0,\qquad \alpha[u]+\lim_{\xi\to\xi_0}(\vec{\omega},\vec{n})_{\mathbb{R}^n}=0,\qquad (t,\xi_0)\in Q_3=\bar{Q}_1\cap\bar{Q}_2.
\end{aligned}$$

We consider a new formulation of the problem where the unknowns are $(u,\vec{\omega})$, and the parabolic problem is converted to a first-order system of partial differential equations with distributional coefficients. We also prove inequalities for negative norms for the parabolic operator with the distributional coefficients and theorems of existence and uniqueness. For optimization problems for the processes we show existence of optimal controls, investigate smoothness of a performance criterion and give a simple condition for controllability of the system. In addition, we consider applications of the obtained results to a pulse control problem and prove convergence of a control mapping regularization procedure.

Keywords: Linear equation; Parabolic equation; Discontinuous solution; Transmission problem; Solvability; Optimization; Distribution; Pulse control

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1. Introduction

There are many actual physical processes occurring in media with foreign zones and insertions. Heat and mass transmission problems often occur in domains with thin low-permeable insertions: paint layers, refractories, gas gaps, thin liquid layers, laminas, cracks, edges of metal granules, etc. When such transmission problems are studied, the foreign zone is eliminated from the domain where heat and mass transmission takes place, and interface (transmission) conditions on the surfaces of insertions are established. Thus, one gets a boundary-value problem in a disconnected domain. There are many papers concerning these problems [1–32], but many problems of solvability and optimization of parabolic systems with discontinuous solutions are still open. This transmission problem admits different standard formulations as evolution variational equality, as Banach-valued time-dependent equation, etc., and there are many references concerning problems of this kind.

Another approach to investigation of heat and mass transmission problems in a domain with thin low-permeable insertions is to replace the original partial differential equation and interface conditions by several first-order partial differential equations which account for the interface conditions themselves [26–30]. In this method, the eliminated insertion is returned to the domain of transmission again, the general equation and transmission conditions turn into a system of first-order partial differential equations, but the coefficients of the equations are now distributions.

In this paper we consider the above approach to the problem. This formulation has some advantages in comparison with the previous ones. In the first-order system, the roles of the variables ξ and t are symmetric. Presence of several equations in the system leaves much more freedom to prove necessary inequalities concerning the operator than there was available in the initial and direct equations. The first-order partial differential equations have simple physical interpretations (they are generalizations of two physical laws: the conservation law and the law of transportation), so that the system is more appropriate for simulating physical processes. In contrast to the formulations as evolution variational equality, where the unknown function u is from a certain space $L_2((0,T);V)$ for V a Banach space, the formulation as the system of equations allows one to study the time-singular processes from the point of view of distribution theory. In particular, this approach is perfectly suited for studying problems of pulse optimal control, and for solving the problem approximately by mixed finite element methods [33,34]. In addition, under this approach, the domain of the process is simply connected (as opposed to those in traditional formulations), which is of importance for some problems (for example, for numerical procedures).

2. Basic definitions

Let the state function $u(t,\xi)$ be defined in a cylindrical domain $Q=(0,T)\times\Omega$, where $t\in(0,T)$, $\xi=(\xi_1,\dots,\xi_n)\in\Omega=\Omega_1\cup\gamma\cup\Omega_2\subset\mathbb{R}^n$, Ω is a bounded simply connected domain with a regular boundary $\partial\Omega$, and $\bar{\gamma}=\bar{\Omega}_1\cap\bar{\Omega}_2\subset\mathbb{R}^n$ is a smooth surface that divides the domain Ω into two simply connected domains Ω_1 and Ω_2 ($\Omega_1\cap\Omega_2=\emptyset$). Denote $Q_i=(0,T)\times\Omega_i$, for $i\in\{1,2\}$, and $Q_3=(0,T)\times\gamma$.