Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 238 (2007) 426–469
Journal of Differential Equations
www.elsevier.com/locate/jde

Global existence and asymptotics of solutions of the Cahn–Hilliard equation

Shuangqian Liu, Fei Wang, Huijiang Zhao *

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

Received 15 November 2006; revised 21 January 2007

Available online 12 March 2007

Dedicated to Professor Xia-Qi Ding on the occasion of his 80th birthday

Abstract

This paper is concerned with the Cauchy problem of the Cahn–Hilliard equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta\varphi(u) + \Delta^2 u = 0, & x \in \mathbf{R}^N,\ t > 0, \\ u|_{t=0} = u_0(x), & x \in \mathbf{R}^N. \end{cases}$$

First, we construct a local smooth solution $u(t,x)$ to the above Cauchy problem, then by combining some a priori estimates, Sobolev's embedding theorem and the continuity argument, the local smooth solution $u(t,x)$ is extended step by step to all $t > 0$ provided that the smooth nonlinear function $\varphi(u)$ satisfies a certain local growth condition at some fixed point $\bar{u} \in \mathbf{R}$ and that $\|u_0(x) - \bar{u}\|_{L^1(\mathbf{R}^N)}$ is suitably small. Secondly, we show that the global smooth solution $u(t,x)$ satisfies the following temporal decay estimates:

$$\left\| D^k\big(u(t,x) - \bar{u}\big)\right\|_{L^p(\mathbf{R}^N)} \leqslant c(\tau)(1+t)^{-\frac{k}{4}-\frac{N}{4}(1-\frac{1}{p})}, \quad t \geqslant \tau > 0,\ k = 0, 1, \ldots.$$

Here $p \in [1, \infty]$, $c(\tau) > 0$ is a constant depending on τ and $\tau > 0$ is any positive constant which can be chosen sufficiently small. At last, we show that, under a strong assumption on the growth of the nonlinear function $\varphi(u)$ at $u = \bar{u}$, the asymptotics of solutions of the above Cauchy problem is described by $\bar{u} + \delta_0 t^{-\frac{N}{4}} G(\frac{x}{\sqrt[4]{t}})$. Here $\delta_0 = \int_{\mathbf{R}^N} (u_0(x) - \bar{u})\,dx$, $G(x) = \int_{\mathbf{R}^N} \exp(-|\eta|^4 + ix\cdot\eta)\,d\eta$.

* Corresponding author.
E-mail address: hhjjzhao@hotmail.com (H. Zhao).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.02.014

Keywords: Cahn–Hilliard equation; Global smooth solution; Optimal temporal decay estimates; Sobolev's embedding theorem; Asymptotics of solutions

Contents

1. Introduction and the statement of our main results

The Cahn–Hilliard (CH) equation

$$\frac{\partial u}{\partial t} + \Delta\varphi(u) + \Delta^2 u = 0, \quad x \in \mathbf{R}^N,\ t > 0,$$

describes phase separation in binary alloys. When such compounds are cooled rapidly to low temperatures below the critical point, they tend to form quickly inhomogeneities forming a granular structure. This phenomenon is called the spinodal decomposition. As a model to describe this phenomenon, CH equation has intrigued many mathematicians' interest and some good results have been obtained (see [1,3,4,8] and references therein). However the presence of the fourth-order differential operator together with the appearance of the nonlinear term $\Delta\varphi(u)$ make its mathematical analysis much difficult than the corresponding second-order equations. Therefore the mathematical results on the CH equation are far from being perfect.

To go directly to the theme of this paper, we only review some former results closely related in the following (a complete list of literatures on the CH equation is beyond the scope of this manuscript, interested authors are referred to [1,3,4,8] and references cited therein): For the one-dimensional case, Charles, M. Ellot and S.M. Zheng studied the following initial–boundary value problem in [1]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \gamma\dfrac{\partial^4 u}{\partial x^4} = \dfrac{\partial^2\varphi(u)}{\partial x^2}, \quad 0 < x < L,\ 0 < t < T, \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(L,t)}{\partial x}, \quad \dfrac{\partial^3 u(0,t)}{\partial x^3} = \dfrac{\partial^3 u(L,t)}{\partial x^3}, \\ u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant L, \\ \varphi(u) = \gamma_2 u^3 + \gamma_1 u^2 - u. \end{cases} \tag{1.1}$$

They have found that the sign of γ_2 in $(1.1)_4$ is crucial: If $\gamma_2 > 0$, there is a unique global smooth solution for the initial–boundary value problem (1.1) for any initial data $u_0 \in H^2(\mathbf{R}, \mathbf{R})$, while if $\gamma_2 < 0$, the solution must blow up in a finite time for large initial data.