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Optimal L^p–L^q convergence rates for the compressible Navier–Stokes equations with potential force ☆

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Abstract

In this paper, we are concerned with the optimal L^p–L^q convergence rates for the compressible Navier–Stokes equations with a potential external force in the whole space. Under the smallness assumption on both the initial perturbation and the external force in some Sobolev spaces, the optimal convergence rates of the solution in L^q-norm with $2 \leqslant q \leqslant 6$ and its first order derivative in L^2-norm are obtained when the initial perturbation is bounded in L^p with $1 \leqslant p < 6/5$. The proof is based on the energy estimates on the solution to the nonlinear problem and some L^p–L^q estimates on the semigroup generated by the corresponding linearized operator.

Keywords: Compressible Navier–Stokes equations; Potential force; Optimal convergence rate; L^p–L^q estimate

1. Introduction

Consider the initial value problem of the compressible Navier–Stokes equations with a potential force in the whole space:

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$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + (u \cdot \nabla) u + \dfrac{\nabla P(\rho)}{\rho} = \dfrac{\mu}{\rho} \Delta u + \dfrac{\mu + \mu'}{\rho} \nabla(\nabla \cdot u) - \nabla \phi(x), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \to (\rho_\infty, 0) \quad \text{as } |x| \to \infty. \end{cases} \tag{1.1}$$

Here, $t > 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The unknown functions $\rho = \rho(t,x) > 0$ and $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ denote the density and velocity, respectively. $P = P(\rho)$ is the pressure function, $-\nabla \phi(x)$ is the time independent potential force, μ, μ' are the viscosity coefficients, and $(\rho_\infty, 0)$ is the state of initial data at infinity. In the following discussion, it is assumed that μ and μ' satisfy the physical conditions $\mu > 0$ and $\mu' + \frac{2}{3}\mu \geqslant 0$, while ρ_∞ is a positive constant and $P(\rho)$ is smooth in a neighborhood of ρ_∞ with $P'(\rho_\infty) > 0$.

For the Navier–Stokes equations $(1.1)_1$–$(1.1)_2$ with potential force, the stationary solution (ρ_*, u_*) is given by $(\rho_*(x), 0)$, where $\rho_*(x)$ satisfies, cf. [14],

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P'(s)}{s} \, ds + \phi(x) = 0. \tag{1.2}$$

The global existence of solutions to the nonlinear problem (1.1) near the steady state $(\rho_*, 0)$ with initial perturbation in H^3 was proved by Matsumura and Nishida [14]. In this paper, we want to obtain the optimal convergence rate of the solution to the steady state when the initial perturbation is also bounded in L^p with $1 \leqslant p < 6/5$. Precisely, the result can be stated as follows.

Theorem 1.1. *Let (ρ, u) be a global classical solution in H^3 to the initial value problem* (1.1), *and $(\rho_*, 0)$ be the corresponding stationary solution. For given $1 \leqslant p < 6/5$, suppose that the potential function $\phi(x)$ and the initial perturbation satisfy*

$$\|\phi\|_{L^{\frac{2p}{2-p}} \cap L^\infty} + \sum_{k=1}^{4} \big\| \big(1 + |x|\big) \nabla^k \phi \big\|_{L^{\frac{2p}{2-p}} \cap L^\infty} \leqslant \epsilon, \tag{1.3}$$

$$\big\| (\rho_0 - \rho_*, u_0) \big\|_{H^3} \leqslant \epsilon, \tag{1.4}$$

for some small constant $\epsilon > 0$, and

$$\big\| (\rho_0 - \rho_*, u_0) \big\|_{L^p} < +\infty. \tag{1.5}$$

Then, there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$, we have

$$\big\| \nabla^k (\rho - \rho_*, u)(t) \big\|_{L^2} \leqslant C_0 (1 + t)^{-\sigma(p,2;1)}, \quad k = 1, 2, 3, \tag{1.6}$$

$$\big\| (\rho - \rho_*, u)(t) \big\|_{L^2} \leqslant C_0 (1 + t)^{-\sigma(p,2;0)}. \tag{1.7}$$

Furthermore,

$$\big\| (\rho - \rho_*, u)(t) \big\|_{L^q} \leqslant C_0 (1 + t)^{-\sigma(p,q;0)}, \quad 2 \leqslant q \leqslant 6. \tag{1.8}$$