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Explicit Hopf–Lax type formulas for Hamilton–Jacobi equations and conservation laws with discontinuous coefficients

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Abstract

We deal with a Hamilton–Jacobi equation with a Hamiltonian that is discontinuous in the space variable. This is closely related to a conservation law with discontinuous flux. Recently, an entropy framework for single conservation laws with discontinuous flux has been developed which is based on the existence of infinitely many stable semigroups of entropy solutions based on an interface connection. In this paper, we characterize these infinite classes of solutions in terms of explicit Hopf–Lax type formulas which are obtained from the viscosity solutions of the corresponding Hamilton–Jacobi equation with discontinuous Hamiltonian. This also allows us to extend the framework of infinitely many classes of solutions to the Hamilton–Jacobi equation and obtain an alternative representation of the entropy solutions for the conservation law. We have considered the case where both the Hamiltonians are convex (concave). Furthermore, we also deal with the less explored case of sign changing coefficients in which one of the Hamiltonians is convex and the other concave. In fact in convex–concave case we cannot expect always an existence of a solution satisfying Rankine–Hugoniot condition across the interface. Therefore the concept of generalised Rankine–Hugoniot condition is introduced and prove existence and uniqueness.

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1. Introduction

We are interested in the following Hamilton–Jacobi equation,

$$\begin{aligned} &v_t + H\big(k(x), v_x\big) = 0, \\ &v(x,0) = v_0(x), \end{aligned} \tag{1}$$

where H is the Hamiltonian and k is a spatially varying and possibly discontinuous coefficient. A special case of (1) is the so called 2-*Hamiltonian* case given by

$$\begin{aligned} &v_t + g(v_x) = 0 \qquad \text{if } x < 0,\ t > 0, \\ &v_t + f(v_x) = 0 \qquad \text{if } x > 0,\ t > 0, \\ &v(x,0) = v_0(x) \qquad \forall x \in \mathbb{R}, \\ &v \in \mathrm{Lip}(\mathbb{R} \times \overline{\mathbb{R}}_+). \end{aligned} \tag{2}$$

In the above case, the Hamiltonian H is discontinuous in the space variable with a single discontinuity at the interface $x = 0$. Equations of the type (1) can arise in several applications like the synthetic aperture radar shape from shading equations in image processing. See [25] for details.

The connections that exist between single conservation laws and Hamilton–Jacobi equations in one space dimension are well known. In particular, let v solve (1), then by taking $u = v_x$, it can be shown that u is a solution of the following single conservation law,

$$\begin{aligned} &u_t + \big(f\big(k(x), u\big)\big)_x = 0, \\ &u(x,0) = u_0(x). \end{aligned} \tag{3}$$

This is an example of a single conservation law with a spatially varying and possibly discontinuous flux function. Similarly (2) is connected to the following conservation law,

$$\begin{aligned} &u_t + \big(H(x)f(u) + \big(1 - H(x)\big)g(u)\big)_x = 0, \\ &u(x,0) = u_0(x). \end{aligned} \tag{4}$$

Equation (4) is a special case of (3) and is called the 2-*flux* case.

Conservation laws with discontinuous flux arise in a wide variety of applications in physics and engineering. To mention a few, they arise while considering two-phase flow in a heterogeneous porous medium that models petroleum reservoir simulation. They also arise while modeling the action of an ideal clarifier–thickener unit that is used in waste-water treatment plants. Details of the applications can be seen in [28].

Hamilton–Jacobi equations with discontinuous Hamiltonian (with a time dependent discontinuous coefficient) were studied by Ostrov in [26]. Under the assumptions that the Hamiltonian is convex (or concave), he used a vanishing viscosity approximation of (1) and passed to the limit in a control formulation to prove existence of viscosity solutions. In [11], Coclite and Risebro studied (1) (with time dependent coefficients and convex fluxes) and obtained existence of