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Nondegeneracy and uniqueness of positive solutions for Robin problem of second order ordinary differential equations and its applications ☆

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Abstract

This article studies positive solutions of Robin problem for semi-linear second order ordinary differential equations. Nondegeneracy and uniqueness results are proven for homogeneous differential equations. Necessary and sufficient conditions for the existence of one or two positive solutions for inhomogeneous differential equations or differential equations with concave–convex nonlinearities are obtained by making use of the nondegeneracy and uniqueness results for positive solutions of homogeneous differential equations.

Keywords: Positive solution; Robin problem; Second order ordinary differential equations

1. Introduction

Let $p, l > 0$, $\alpha, \beta, \gamma, \eta \geqslant 0$ be real numbers. We consider the following problem:

$$\begin{cases} u'' + u^p = 0, & x \in (-l, l), \\ u > 0, & x \in (-l, l), \\ \alpha u'(-l) - \beta u(-l) = 0, \\ \gamma u'(l) + \eta u(l) = 0. \end{cases} \tag{1.1}$$

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The boundary conditions in problem (1.1) are called Robin boundary conditions. Such Robin boundary conditions are particularly interesting in biological models where they often arise. We refer the reader to [6] for this aspect. The question of the existence of solutions of problem (1.1) has been extensively treated in various literature, see for example [3,8] and the references cited therein, and will not be considered here. The main purpose of the present paper is to study the qualitative properties of solutions of problem (1.1) and its applications. The properties of solutions of problem (1.1) which we are interested in here include the nondegeneracy (see the definition below), the uniqueness, the a priori estimate and so on. A nondegeneracy result for solutions of problem (1.1) when $\alpha = \gamma = 0$ was proved in [13], and there are no nondegeneracy results for solutions of problem (1.1) in any other cases. Uniqueness results for solutions of problem (1.1) when $\alpha = \gamma = 0$, or $\alpha = \eta = 0$ were proved in [5,12]. In many literature, the most powerful tool used to prove uniqueness result for Dirichlet and Dirichlet–Neumann problem of second order ordinary differential equations is the so-called time mapping method which is developed in [5,12]. In the concluding remarks of [12], Laetsch claimed that the time mapping method can be used to prove uniqueness results for the general problem (1.1). However, rigorous proof had not been given in that paper. In recent paper [4], V. Anuradha, C. Maya and R. Shivaji gave a detailed proof of Laetsch's claim for Neumann–Robin problem, that is the problem (1.1) with respect to $\beta = 0$. Moreover, we would like to point out that time mapping method also works well for symmetric Robin problem, that is the problem (1.1) with respect to $\alpha = \gamma \neq 0$ and $\beta = \eta \neq 0$. For a similar argument, we refer to [9,15]. However, in our point of view, the time mapping method is difficult to be applied in the case of $\alpha, \beta, \gamma, \eta > 0$ and $\alpha \neq \gamma$, or $\alpha, \beta, \gamma, \eta > 0$ and $\beta \neq \eta$. In fact, if we choose $u(-l) = m$ as a time parameter as in [4] and try to prove an uniqueness result for problem (1.1) by making use of time mapping method, then for any given l, we will face to prove the following algebraic system of m and $u(l)$ have exactly one solution pair $(m, u(l))$:

$$\begin{cases} 2l = \displaystyle\int_m^c \frac{dv}{\sqrt{(\frac{\beta}{\alpha})^2 m^2 + \frac{2}{p+1} m^{p+1} - \frac{2}{p+1} v^{p+1}}} + \int_{u(l)}^c \frac{dv}{\sqrt{(\frac{\beta}{\alpha})^2 m^2 + \frac{2}{p+1} m^{p-1} - \frac{2}{p+1} v^{p-1}}}, \\ \left(\dfrac{\eta}{\gamma}\right)^2 u^2(l) + \dfrac{2}{p+1} u^{p+1}(l) = \left(\dfrac{\beta}{\alpha}\right)^2 m^2 + \dfrac{2}{p+1} m^{p+1}, \end{cases}$$

where $c = [\frac{p+1}{2}(\frac{\beta}{\alpha})^2 m^2 + m^{p+1}]^{1/(p+1)}$. This seems to be a difficult problem.

In this paper, we will adopt a new scheme to prove an uniqueness result for problem (1.1) when $\alpha, \beta, \gamma, \eta > 0$. In our scheme, a crucial step is to prove that solutions of problem (1.1) when $\alpha, \beta, \gamma, \eta > 0$ are nondegenerate. Here, the nondegeneracy of solutions of problem (1.1) is defined as follows:

Definition. Any given solution $u(x)$ of problem (1.1) is called nondegenerate if the following linearized problem

$$\begin{cases} \psi'' + p u^{p-1} \psi = 0, \quad x \in (-l, l), \\ \alpha \psi'(-l) - \beta \psi(-l) = 0, \\ \gamma \psi'(l) + \eta \psi(l) = 0 \end{cases} \tag{1.2}$$

admits only the trivial solution $\psi \equiv 0$ in $[-l, l]$.