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Inverse spectral problems for Bessel operators

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Abstract

We study the inverse spectral problem for a class of Bessel operators given in $L_2(0,1)$ by the differential expression

$$-\left(\frac{d}{dx}-\frac{\kappa}{x}-v\right)\left(\frac{d}{dx}+\frac{\kappa}{x}+v\right)$$

with $\kappa \in \mathbb{N}$ and a real-valued function $v \in L_p(0,1)$, $p \in [1,\infty)$, subject to various boundary conditions. We describe completely the spectral data of these operators, i.e., the spectra and corresponding norming constants, and give the algorithm of reconstruction of v from the spectral data.

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Keywords: Inverse spectral problem; Sturm–Liouville equation; Bessel equation; Double commutation method

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1. Introduction

The main aim of the paper is to solve the inverse spectral problem for a class of general Bessel operators on the interval [0, 1] generated by the differential expression (1.3) below. Such operators naturally appear in the spectral analysis of the radial Schrödinger operators $-\Delta + q(|\mathbf{x}|)$ acting on a unit ball of $\mathbb{R}^3$. Indeed, it is well known that the latter can be decomposed in spherical coordinates into the direct sum of operators T_κ parametrized by the *angular momentum* $\kappa \in \mathbb{Z}_+$ and given by

$$T_\kappa = -\frac{d^2}{dx^2} + \frac{\kappa(\kappa+1)}{x^2} + q(x).$$

For real-valued $q \in L_2(0,1)$ and $\kappa \geqslant 1$, the operator T_κ becomes self-adjoint under the boundary condition

$$y(1)\cos\theta = y'(1)\sin\theta, \quad \theta \in [0,\pi),$$

and is then denoted by $T(\theta,\kappa,q)$. This operator is usually called the Bessel operator since for $q \equiv 0$ its eigenfunctions are expressed in terms of Bessel functions, see [32]. When $\kappa = 0$, $T(\theta,0,q)$ is given by a regular Sturm–Liouville differential expression, and another boundary condition is needed to get a self-adjoint operator; for $\eta \in [0,\pi)$, we denote by $T_\eta(\theta,0,q)$ the restriction of $T(\theta,0,q)$ by the boundary condition

$$y(0)\cos\eta = y'(0)\sin\eta.$$

The operator $T(\theta,\kappa,q)$ is bounded below and has a simple discrete spectrum. It was shown in [7] that for $q \in L_2(0,1)$ the eigenvalues $\lambda_n^2(\theta,\kappa,q)$ of $T(\theta,\kappa,q)$, when arranged in increasing order, have the following asymptotics:

$$\lambda_n^2(0,\kappa,q) = \pi^2\left(n + \frac{\kappa}{2}\right)^2 + s_0 + r_n(0), \tag{1.1}$$

$$\lambda_n^2(\theta,\kappa,q) = \pi^2\left(n + \frac{\kappa-1}{2}\right)^2 + s_\theta + r_n(\theta), \quad \theta \neq 0, \tag{1.2}$$

where s_θ are real constants (in particular, $s_0 = \int_0^1 q\,dx - \kappa(\kappa+1))$ and the sequences $(r_n(\theta))$ belong to $\ell_2(\mathbb{N})$. We observe that the nonzero momentum κ shifts the main asymptotic term of λ_n by $\kappa/2$, while the remainders are as for the case $\kappa = 0$. Loosely speaking, the operator $T(\theta,\kappa,q)$ has $\kappa/2$ eigenvalues less than a regular Sturm–Liouville operator.

For the case $\kappa = 0$, there exists a detailed inverse spectral theory, cf. [15,22–25]. In particular, Pöschel and Trubowitz in [25] studied in detail the mapping sending the potential q into the corresponding spectral data—the sequences of eigenvalues and the corresponding norming constants—and proved that this mapping is one-to-one and analytic.

In [19] the method of [25] was extended to the case $\kappa = 1$, allowing a complete description of the possible spectra of the operators $T(0,1,q)$ when q runs through $L_{2,\mathbb{R}}(0,1)$, the real space of real-valued functions in $L_2(0,1)$, and the structure of the corresponding isospectral manifold. Recently, Serier [29] extended this analysis to the case of an arbitrary $\kappa \in \mathbb{N}$.