Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 234 (2007) 391–411
Journal of Differential Equations
www.elsevier.com/locate/jde

# An analysis of phase noise and Fokker–Planck equations ☆

Shui-Nee Chow, Hao-Min Zhou *

*School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*

Received 20 October 2005

**Abstract**

A local moving orthonormal transformation has been introduced to rigorously study phase noise in stochastic differential equations (SDEs) arising from nonlinear oscillators. A general theory of phase and amplitude noise equations and its corresponding Fokker–Planck equations are derived to characterize the dynamics of phase and amplitude error. As an example, a van der Pol oscillator is considered by using the general theory.


## 1. Introduction

Phase noise in nonlinear oscillators is very important in circuit design and other areas such as optics. For example, it is known that timing jitter in circuit design is caused by phase noise [9,15]. Mathematically, nonlinear oscillators can often be described by nonlinear autonomous differential equations with periodic orbits (limit cycles in the plane) that are orbitally asymptotically stable. We note that any solutions near an orbitally asymptotically stable periodic orbit in phase space will stay close to the periodic orbit and approach the periodic orbit in phase space with asymptotic phase [7]. However, noise is unavoidable in practice and is often modeled by additional stochastic terms in the nonlinear differential equations. In Fig. 1, we have an asymptotically stable periodic orbit $\Gamma$ (solid line) in phase space with least period $T > 0$ of an unperturbed nonlinear oscillator. The orbit returns to its initial state after time $T$. However, a perturbed solu-

☆ Research supported in part by grants NSF DMS-0410062.
* Corresponding author.
*E-mail addresses:* chow@math.gatech.edu (S.-N. Chow), hmzhou@math.gatech.edu (H.-M. Zhou).

0022-0396/$ – see front matter 
doi:10.1016/j.jde.2006.11.015

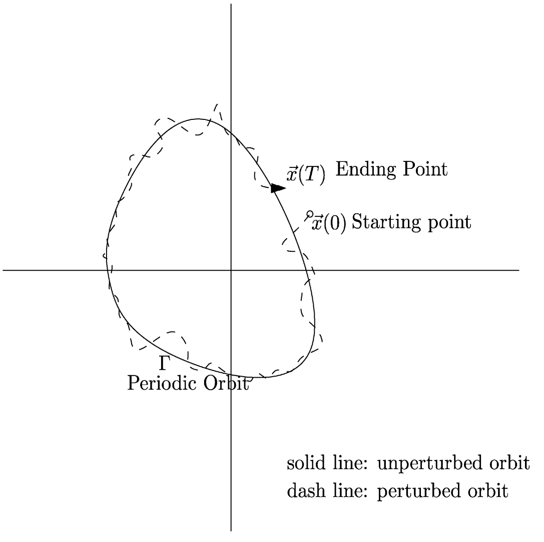


Fig. 1. Perturbations near an orbitally asymptotically stable limit cycle $\Gamma$. Solutions will not return to their starting states after period $T$.

tion does not return to the starting point after the same time $T$ due to random perturbations. Thus, natural rhythm of the oscillator is disturbed. Phase noise refers to the variations in the oscillation frequency, and jitter is the fluctuations in the period.

There is a large literature dealing with phase noise problems (see, for example, [4,5,10,13,16, 17] and references therein). However, it is indicated in [4], that theoretical understanding in the subject is rather incomplete. The main difficulty is how to completely separate phase and amplitude components in the error analysis in the nonlinear dynamics under random perturbations, which is the goal of this paper.

Standard approaches to study phase noise are largely based on linearizations of the nonlinear dynamic systems. The main idea is to use linear parts in Taylor expansions to replace the nonlinear terms near the unperturbed orbits. The key assumption for this idea to be useful is that the difference between perturbed and unperturbed solutions remains small. However it has been discussed in both [4,11] that the deviation of the perturbed solution from the unperturbed solution can grow to infinitely large even for orbits that are orbitally exponentially asymptotically stable. This is the reason that why linearization strategies can lead to incorrect characterization of the real phenomena in phase noise analysis.

Recently, two different nonlinear approaches have been proposed. One is based on Floquet theory and by considering a delay phase coordinate to characterize the leading contributions of the phase noise [4]. The delay phase coordinate satisfies a stochastic differential equation depending on the largest eigenvalue (must be 1 to sustain the periodic orbit) of the transition (monodromy) matrix of the linearized system and its corresponding eigenfunction. Phase noise from other components of spectrum of the transition matrix decays to zero eventually if one assumes that the random perturbations exist for only a finite time of period.

The second approach is based on the Fokker–Planck equation associated with the SDE. The standard SDE theory suggests that every diffusive SDE including the SDE governing the oscillator considered in this paper corresponds to a parabolic equation (Fokker–Planck equation, also called Kolmogorov equation in many literature) which is used to describe the evolution of the