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ScienceDirect
J. Differential Equations 234 (2007) 607–625
Journal of Differential Equations
www.elsevier.com/locate/jde

The stability of attractors for non-autonomous perturbations of gradient-like systems

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Received 12 September 2006; revised 17 November 2006

Available online 19 December 2006

Abstract

We study the stability of attractors under non-autonomous perturbations that are uniformly small in time. While in general the pullback attractors for the non-autonomous problems converge towards the autonomous attractor only in the Hausdorff semi-distance (upper semicontinuity), the assumption that the autonomous attractor has a 'gradient-like' structure (the union of the unstable manifolds of a finite number of hyperbolic equilibria) implies convergence (i.e. also lower semicontinuity) provided that the local unstable manifolds perturb continuously.

We go further when the underlying autonomous system is itself gradient-like, and show that all trajectories converge to one of the hyperbolic trajectories as $t \to \infty$. In finite-dimensional systems, in which we can reverse time and apply similar arguments to deduce that all bounded orbits converge to a hyperbolic trajectory as $t \to -\infty$, this implies that the 'gradient-like' structure of the attractor is also preserved under small non-autonomous perturbations: the pullback attractor is given as the union of the unstable manifolds of a finite number of hyperbolic trajectories.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.11.016

1. Introduction

In autonomous systems the theory of global attractors is well-developed for both ordinary and partial differential equations (e.g. [6,8,17,18]). Nevertheless, there are only detailed results on the structure of such attractors for gradient-like systems: in this case the attractors are formed from the union of the unstable manifolds of the equilibrium points. While results on the upper semi-continuity of attractors under perturbation (no 'explosion') hold for a wide class of equations, these gradient-like examples are the only systems for which full continuity results are available.

Here we show that similar results hold even when the perturbations are non-autonomous. The key assumption is that the local stable and unstable manifolds of hyperbolic equilibria perturb in a smooth way, and we present our main results in an abstract form which we believe serves to keep both the hypotheses and the arguments clearer than they would be in particular examples.

In gradient-like systems every trajectory tends to one of the equilibria. Ball and Peletier [1] showed that a similar result holds for systems that are asymptotically autonomous, with a limit system that is gradient-like. Here we show a similar result for small non-autonomous perturbations of gradient like systems, namely that all solutions tend to distinguished hyperbolic trajectories corresponding to the equilibria of the unperturbed system. Ball and Peletier's result is then a corollary of ours.

In finite-dimensional systems one can reverse the sense of time. It follows in this case that every trajectory defined for all time also tends to one of these hyperbolic trajectories as $t \to -\infty$. In this situation we can show that the structure of the autonomous attractor is also preserved under small non-autonomous perturbations: the pullback attractor is the union of the unstable manifolds of the hyperbolic trajectories.

To end the paper we discuss the application of our results to finite and infinite-dimensional semilinear equations on Banach spaces, making use of recent results on the stability on local stable and unstable manifolds due to Carvalho and Langa [3].

1.1. Standing assumptions

Throughout the paper we will assume that all of the conditions in this section are satisfied.

Let $\mathscr{B}$ be a Banach space with norm $\|\cdot\|$. Suppose that we have an underlying autonomous dynamical system $\{S_0(t)\}_{t\geqslant 0}$ defined on $\mathscr{B}$, where

$$\lim_{t\downarrow 0} S_0(t)x = S_0(0)x = x, \quad x \in \mathscr{B}, \qquad S_0(t+s) = S_0(t)S_0(s) \quad \text{for all } t, s \geqslant 0,$$

and for each $t \geqslant 0$ the operator $S_0(t)$ is continuous from $\mathscr{B}$ into $\mathscr{B}$. We assume that this system has a global attractor $\mathcal{A}_0$, i.e. a compact invariant set that attracts all bounded subsets X of $\mathscr{B}$,

$$\operatorname{dist}\big(S_0(t)X, \mathcal{A}_0\big) \to 0 \quad \text{as } t \to \infty,$$

where

$$\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|.$$