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Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms

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Abstract

Based on new information concerning strongly indefinite functionals without Palais–Smale conditions, we study existence and multiplicity of solutions of the Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where V and g are periodic with respect to x and 0 lies in a gap of $\sigma(-\Delta + V)$. Supposing g is asymptotically linear as $|u| \to \infty$ and symmetric in u, we obtain infinitely many geometrically distinct solutions. We also consider the situation where g is super linear with mild assumptions different from those studied previously, and establish the existence and multiplicity.

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Keywords: Schrödinger equation; Indefinite; Asymptotically linear; Infinitely many solutions

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1. Introduction and main results

Consider the following Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{NS}$$

where V and g are continuous real functions and satisfy

(V_0) $V(x)$ is 1-periodic in x_j for $j = 1, \dots, N$ such that $0 \notin \sigma(-\Delta + V)$;
(N_0) $g(x,u)$ is 1-periodic in x_j for $j = 1, \dots, N$, $G(x,u) \geqslant 0$ and $g(x,u) = o(|u|)$ as $u \to 0$ uniformly in x.

In this paper we are interested in existence of infinitely many geometrically distinct solutions of (NS) when the problem is strongly indefinite, that is, 0 lies in a gap of the spectrum $\sigma(A)$, $A := -\Delta + V$, and $g(x,u)$ is of asymptotically linear growth as $|u| \to \infty$. As far as we are aware there were no such multiplicity results in this situation. We also deal with the case where $g(x,u)$ is of superlinear growth as $|u| \to \infty$ with conditions different from those studied deeply in previous related works.

The Schrödinger equation with periodic potentials and nonlinearities has found a great deal of interest in last years because not only it is important in applications but it provides a good model for developing mathematical methods, see, e.g., [1–3,5, 7,9–11, 13–18,22,23,25] and the references therein. It is known that for periodic potentials $\sigma(A)$ is a union of closed intervals (cf. [20]). There have been many results on existence and multiplicity of solutions of such an equation depending on the location of 0 in $\sigma(A)$, among which we recall the following ones.

Case 1: $0 < \inf \sigma(A)$. In [11] Coti-Zelati and Rabinowitz proved via a mountain-pass argument that (NS) has infinitely many solutions provided $g \in \mathcal{C}^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and satisfies *the superlinear condition: there is* $\mu > 2$ *such that*

$$0 < \mu G(x,u) \leqslant g(x,u)u \quad \text{for all } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R} \setminus \{0\} \tag{1.1}$$

and the subcritical condition: there is $s \in (2, 2^*)$ *such that*

$$|g_u(x,u)| \leqslant c_1 + c_2|u|^{s-2} \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.2}$$

Here (and in the following) $G(x,u) := \int_0^u g(x,t)\,dt$, $2^* = \infty$ if $N = 1, 2$, $2^* = 2N/(N-2)$ if $N \geqslant 3$, and c_i denote positive constants. This result was shown recently in [14,23] to remain true for more general nonlinearities, particularly, for asymptotically linear ones.

Case 2: 0 *lies in a gap of* $\sigma(A)$, *that is*,

$$\underline{\Lambda} := \sup\left(\sigma(A) \cap (-\infty, 0)\right) < 0 < \overline{\Lambda} := \inf\left(\sigma(A) \cap (0, \infty)\right). \tag{1.3}$$