Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 240 (2007) 249–278
Journal of Differential Equations
www.elsevier.com/locate/jde

Weak and strong attractors for the 3D Navier–Stokes system

A.V. Kapustyan [a], J. Valero [b,*]

[a] *Kiev National Taras Shevchenko University, 01033 Kiev, Ukraine*
[b] *Universidad Miguel Hernández, Centro de Investigación Operativa, Avda. Universidad, s/n., Elche (Alicante), 03202, Spain*

Received 11 August 2005; revised 17 May 2007

Available online 27 June 2007

Abstract

We study in this paper the asymptotic behaviour of the weak solutions of the three-dimensional Navier–Stokes equations. On the one hand, using the weak topology of the usual phase space H (of square integrable divergence free functions) we prove the existence of a weak attractor in both autonomous and nonautonomous cases. On the other, we obtain a conditional result about the existence of the strong attractor, which is valid under an unproved hypothesis. Also, with this hypothesis we obtain continuous weak solutions with respect to the strong topology of H.

MSC: 35B40; 35B41; 35K55; 35Q30; 37B25; 58C06

Keywords: Three-dimensional Navier–Stokes equations; Set-valued dynamical system; Global attractor

1. Introduction

There has been in the last years, and also in the present a great interest in studying the asymptotic behaviour of the weak solutions of the three-dimensional Navier–Stokes equations. Whereas in the two-dimensional case the existence of the global attractor is a well-known result in both the autonomous and nonautonomous cases (see [4,10,22,31,35]), the three-dimensional case contains some difficult problems to overcome. On the one hand, it is not known whether

* Corresponding author.
E-mail addresses: alexkap@univ.kiev.ua (A.V. Kapustyan), jvalero@umh.es (J. Valero).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.06.008

the weak solution corresponding to the Cauchy problem is unique or not. On the other, and this is the main difficulty, so far the weak solutions were proved to be continuous in time only with respect to the weak topology of the phase space.

Several partial results have been obtained so far by different authors. Raugel and Sell [29] proved the existence of the attractor in thin domains, whereas in [11,33] it is studied the existence of a trajectory attractor. The last result was generalized to the stochastic case by Flandoli and Schmalfuss [15]. The main idea in this method is to replace the usual phase space by the space of all trajectories. Then it is studied the asymptotic behaviour of the translation semigroup. Another approach, which is similar to the previous one, is used in [13], where instead of the whole trajectory it is studied the asymptotic behaviour of small pieces of them, that is, of the restriction of the solution to a small time interval. The idea in these two methods is to avoid the problem of the lack of continuity by using a weaker topology, namely, the topology of square integrable functions on finite intervals of time.

As mentioned before, the main difficulty in trying to prove the existence of the global attractor is the lack of continuity in time of the weak solutions. Assuming the unproved hypothesis that the weak solutions are continuous in the strong topology, Ball [5] obtained the existence of the global attractor. Later on, this result was extended in [12] and [32]. The stochastic case is considered in [28].

Finally, in [6,7] the authors obtain the existence of the attractor assuming strong restrictions on the external force. The idea in these papers is that, after a sufficiently big time, the weak solutions become strong ones, and then a standard continuous semigroup can be defined.

Our first aim in this paper is related to the existence of a strong global attractor. As in the above mentioned papers, we have obtained only a conditional result, that is, valid under an unproved hypothesis on the solutions.

Assuming that for every initial data in V there exists a weak solution satisfying a suitable estimate in the space $(L^4(\Omega))^3$ we prove the existence of at least one strongly continuous solution for every initial data in H. The same result is obtained if we assume that the Navier–Stokes system is well posed in the space V, where H, V are the usual spaces of the Navier–Stokes system.

Once the problem of the continuity is solved (conditionally), we use the method of multivalued semiflows or processes (see [2,3,24–26]) in order to study the asymptotic behaviour of the solutions; in particular, we obtain the existence of a global compact attractor in both autonomous and nonautonomous cases. Another approach, which is rather similar, is the method of generalized semigroups (see [5,14]). A comparison of these two theories can be found in [8]. The method of multivalued semiflows has been applied successfully in several models (see, among others, [9,19,20,27,36]).

We note that it is well known that when the Navier–Stokes system is well posed in V, then the corresponding semigroup in V possesses a compact global attractor (see [35, p. 382]). Now, we have proved that if the Navier–Stokes system is well posed in V, then we can define a multivalued semiflow in H having a compact global attractor.

Our second aim is to prove the existence of a weak global attractor. For external forces f in $L^\infty(\mathbb{R}; H)$ we define a family of multivalued processes U^R from the ball of radius $R \geqslant R_0$ into itself, where R_0 is a fixed constant depending on the parameters of the problem. We prove for any $R \geqslant R_0$ the existence of a global attractor $\mathcal{A}_R$ but considering the attracting property in the weak topology of the phase space. Moreover, it is shown that the global attractor does not depend on R, i.e., $\mathcal{A}_R = \mathcal{A}_{R_0}$, for all $R \geqslant R_0$. As a particular case, we obtain these results in the autonomous case, i.e., when $f \in H$.