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Stability theory and Lyapunov regularity ☆

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Abstract

We establish the stability under perturbations of the dynamics defined by a sequence of linear maps that may exhibit both *nonuniform* exponential contraction and expansion. This means that the constants determining the exponential behavior may increase exponentially as time approaches infinity. In particular, we establish the stability under perturbations of a nonuniform exponential contraction under appropriate conditions that are much more general than uniform asymptotic stability. The conditions are expressed in terms of the so-called regularity coefficient, which is an essential element of the theory of Lyapunov regularity developed by Lyapunov himself. We also obtain sharp lower and upper bounds for the regularity coefficient, thus allowing the application of our results to many concrete dynamics. It turns out that, using the theory of Lyapunov regularity, we can show that the *nonuniform* exponential behavior is ubiquitous, contrarily to what happens with the uniform exponential behavior that although robust is much less common. We also consider the case of infinite-dimensional systems.

MSC: 37C75

Keywords: Lyapunov regularity; Nonuniform exponential contractions; Nonuniform exponential dichotomies

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1. Introduction

The main theme of our paper is the relation between the stability theory of dynamical systems and the so-called Lyapunov regularity theory. In particular, we are interested in establishing the stability under perturbations of *nonuniform* exponential contractions and *nonuniform* exponential dichotomies. We are mainly interested in the case of perturbations of a nonautonomous dynamics defined by a sequence of linear maps. We consider both finite-dimensional and infinite-dimensional systems.

In particular, a nonuniform exponential contraction allows a "spoiling," possibly exponential, of the uniform contraction along the solution as the initial time increases. Thus, even though we can still establish the exponential stability of the solutions, in the nonuniform case the size of the neighborhood in which we must choose the initial condition so that the corresponding solution satisfies a prescribed bound may depend on the initial time. We recall that in the uniform case this neighborhood can be chosen independently of the initial time. In a similar manner, the notion of nonuniform exponential dichotomy imitates the classical notion of (uniform) exponential dichotomy, although in the nonuniform case we allow a "spoiling," again possibly exponential, of the uniform contraction and uniform expansion along the solution as the initial time approaches infinity. The notions are recalled in the main text. We refer to the book [1] for an introduction to the theory of nonuniformly hyperbolic dynamics.

It should be emphasized that the notions of uniform exponential contraction and uniform exponential dichotomy demand considerably from the dynamics. Of course that there exist large classes of dynamical systems possessing this exponential behavior, and even more this class is robust under sufficiently small perturbations. For a detailed discussion, references, and historical comments, we strongly recommend the book [4]. See also [5–9]. On the other hand, we can show that, using the so-called Lyapunov regularity theory, the notions of nonuniform exponential contraction and nonuniform exponential dichotomy are very common. Indeed, *any* linear nonautonomous dynamics possessing only negative Lyapunov exponents admits a nonuniform exponential contraction, and essentially *any* linear nonautonomous dynamics possessing both negative and nonnegative Lyapunov exponents admits a nonuniform exponential dichotomy. This indicates that the nonuniform exponential behavior is very common, and in fact much more common than the uniform exponential behavior. Thus, it is quite reasonable to study the stability under perturbations in the nonuniform setting.

In particular, we establish the stability under perturbations of the nonautonomous dynamics defined by a sequence of linear maps, under appropriate conditions that are much more general than uniform asymptotic stability. Roughly speaking, these conditions ensure that the "nonuniformity" of the exponential behavior is sufficiently small when compared to the nonlinear perturbation. Fortunately, the theory of Lyapunov regularity already possesses an invariant that can be used to express the above conditions on the smallness of the "nonuniformity" of the exponential behavior. Namely, this is the so-called regularity coefficient. Thus, having in mind the application of the stability results to concrete systems, it is crucial to obtain sharp estimates for the regularity coefficient, that hopefully can be given somewhat explicitly in terms of the linear unperturbed dynamics. In particular, we give sharp lower and upper bounds for the regularity coefficient. In addition, we give several alternative characterizations of the situation when the dynamics is Lyapunov regular, i.e., when the regularity coefficient is zero. In this latter case, we essentially can consider any perturbation in our stability results.