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Minimal periods of semilinear evolution equations with Lipschitz nonlinearity

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Abstract

It is known that any periodic orbit of a Lipschitz ordinary differential equation $\dot{x} = f(x)$ must have period at least $2\pi/L$, where L is the Lipschitz constant of f. In this paper, we prove a similar result for the semilinear evolution equation $du/dt = -Au + f(u)$: for each α with $0 \leqslant \alpha \leqslant 1/2$ there exists a constant K_α such that if L is the Lipschitz constant of f as a map from $D(A^\alpha)$ into H then any periodic orbit has period at least $K_\alpha L^{-1/(1-\alpha)}$. As a concrete application we recover a result of Kukavica giving a lower bound on the period for the 2d Navier–Stokes equations with periodic boundary conditions.

Keywords: Period orbits; Minimal period; Semilinear evolution equations; Navier–Stokes equations

1. Introduction

Yorke [12] showed that any periodic orbit of an ordinary differential equation $\dot{x} = f(x)$ must have period at least $2\pi/L$, where L is the (global) Lipschitz constant of f, i.e.

$$|f(x) - f(y)| \leqslant L|x - y|.$$

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As well as being interesting in its own right, this result is useful since it allows one to show that the conditions required by the Takens time-delay embedding theorem are satisfied provided that the time delay is taken sufficiently small (see [9] for a proof of this theorem in the ODE case).

Recently Robinson [8] has proved a version of the Takens embedding theorem valid for infinite-dimensional systems, and so a similar result guaranteeing a minimum period would be useful in this context, as well as once again being of independent interest.

Although there is no general framework that will encompass all possible PDEs, the semilinear evolution equations studied by Henry [4] are general enough to include reaction–diffusion equations and the Navier–Stokes equations. Here, we prove that any periodic orbit of the equation

$$\frac{du}{dt} = -Au + f(u),$$

where A is a positive self-adjoint operator and f has Lipschitz constant L from $D(A^{\alpha})$ into H for $0 \leqslant \alpha \leqslant 1/2$, must have period at least $K_{\alpha} T^{-1/(1-\alpha)}$, where K_{α} depends only on α.

Our argument is inspired in part by that of Kukavica [5], who exploited the time analyticity of solutions of the Navier–Stokes equations to show that there is a lower bound on the period of any periodic orbit, even for the three-dimensional case (where existence and uniqueness results are not available in general).

In Section 2 we give a simple proof of the ODE result, and the in Section 3 we give the new result for semilinear evolution equations. The final section discusses the application of the result to the 2d Navier–Stokes equations, illustrating the techniques available for equations that possess a global attractor.

2. Lipschitz ODEs

In this section we give a simple proof of the result for ODEs, following ideas in [5]. As well as being more straightforward than the proof of Yorke [12], this also serves as a taster for the more involved argument in the next section.

Theorem 2.1. *Any periodic orbit of the equation $\dot{x} = f(x)$, where f has Lipschitz constant L, has period $T \geqslant 1/L$.*

As remarked in the introduction, Yorke [12] showed that the period is in fact bounded below by $2\pi/L$.

Proof of Theorem 2.1. Fix $\tau > 0$ and set $v(t) = x(t) - x(t-\tau)$. Then

$$v(t) - v(s) = \int_s^t \dot{v}(r)\, dr.$$