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Three spheres inequalities for a two-dimensional elliptic system and its application

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Abstract

In this paper we prove three spheres inequalities for a two-dimensional strongly elliptic system. We then give an application of these three spheres inequalities to the inverse problem of identifying cavities by partial boundary measurements.

1. Introduction

Assume that Ω, a bounded open set in $\mathbb{R}^2$ with C^2 boundary. Without loss of generality, we may assume that Ω contains the origin and $\overline{B_R} \subset \Omega$ for some $R > 0$. We denote B_R the open ball centered at the origin with radius R. Let $C(x) = (C_{ijkl}(x)) \in W^{1,\infty}(\Omega)$ be a real rank-four tensor satisfying the major symmetry property

$$C_{ijkl}(x) = C_{klij}(x) \quad \forall i, j, k, l, \text{ and } x \in \Omega, \tag{1.1}$$

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and the strong ellipticity condition, i.e. there exists a constant $\delta > 0$ such that

$$\sum_{ijkl} C_{ijkl}(x) a_i b_j a_k b_l \geqslant \delta |a|^2 |b|^2 \tag{1.2}$$

for all $x \in \Omega$ and vectors $a = [a_1, a_2]^\top$, $b = [b_1, b_2]^\top$. Here and below, all Latin indices are set to be from 1 to 2. We consider the equilibrium equation

$$(\mathcal{L}u)_i := \sum_{j,k,l} C_{ijkl} \partial_j \partial_l u_k + \sum_{kl} A_{ikl} \partial_l u_k + \sum_k B_{ik} u_k = 0 \quad \text{in } \Omega \tag{1.3}$$

with $A_{ikl}(x), B_{ijkl}(x) \in L^\infty(\Omega)$. A special case of (1.3) is the two-dimensional elasticity system.

To state our main result, we now rewrite (1.3) into the matrix form with $u = [u_1, u_2]^\top$, namely,

$$\mathcal{L}u = \Lambda_{11} \partial_1^2 u + \Lambda_{12} \partial_1 \partial_2 u + \Lambda_{22} \partial_2^2 u + \mathcal{R}u = 0, \tag{1.4}$$

where

$$\Lambda_{11} = (C_{i1k1}), \qquad \Lambda_{22} = (C_{i2k2}), \qquad \Lambda_{12} = \Phi + \Phi^t \quad \text{with } \Phi = (C_{i2k1})$$

and

$$\mathcal{R}u = A_1 \partial_1 u + A_2 \partial_2 u + Bu, \quad \text{with } A_j = (A_{ikj}) \text{ and } B = (B_{ik}).$$

The main result of the paper is now stated as follows.

Theorem 1.1. *Assume that the tensor $C(x) = (C_{ijkl}(x)) \in W^{1,\infty}(\Omega)$ satisfies* (1.1) *and* (1.2). *Let $(\tilde{\mu}(x), z(x)) \in W^{1,\infty}(\Omega)$ be an eigenpair of the quadratic pencil $\Lambda_{11}\lambda^2 + \Lambda_{12}\lambda + \Lambda_{22}$, i.e.*

$$\left(\Lambda_{11}\tilde{\mu}^2 + \Lambda_{12}\tilde{\mu} + \Lambda_{22}\right) z(x) = 0 \quad \forall x \in \Omega.$$

Furthermore, we suppose that the matrix function $[z(x), \bar{z}(x)]$ is nonsingular for all $x \in \Omega$. Then for R_1, R_2, and R_3 satisfying $0 < R_1 < R_2 < R_3 \leqslant R$, there exist positive constants $c > 0$ and $0 < \tau < 1$ such that

$$\int_{B_{R_2}} |u|^2\, dx \leqslant c \left(\int_{B_{R_1}} |u|^2\, dx \right)^{\tau} \left(\int_{B_{R_3}} |u|^2\, dx \right)^{1-\tau} \tag{1.5}$$

for $u \in H^1(B_R)$ satisfying (1.3) *in B_R, where c and τ depend on R_1/R_3, R_2/R_3, and coefficients C_{ijkl}, A_{jkl}, B_{ik}.*

Remark 1.2. It was shown in [15] that the assumptions on the eigenpair of the quadratic pencil $\Lambda_{11}\lambda^2 + \Lambda_{12}\lambda + \Lambda_{22}$ is generic. Furthermore, if $\mathcal{L}u$ is the isotropic elasticity system, then these assumptions hold without extra restriction on Lamé coefficients (see [13]).