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New solutions for nonlinear Schrödinger equations with critical nonlinearity

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Abstract

We consider the following nonlinear Schrödinger equations in $\mathbb{R}^n$

$$\begin{cases} \varepsilon^2 \Delta u - V(r)u + u^p = 0 & \text{in } \mathbb{R}^n; \\ u > 0 & \text{in } \mathbb{R}^n \text{ and } u \in H^1(\mathbb{R}^n), \end{cases}$$

where $V(r)$ is a radially symmetric positive function. In [A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres, Part I, Comm. Math. Phys. 235 (2003) 427–466], Ambrosetti, Malchiodi and Ni proved that if $M(r) = r^{n-1}(V(r))^{\frac{p+1}{p-1} - \frac{1}{2}}$ has a nondegenerate critical point $r_0 \neq 0$, then a layered solution concentrating near r_0 exists. In this paper, we show that if $p = \frac{n+2}{n-2}$ and the dimension $n = 3, 4$ or 5, another new type of solution exists: this solution has a layer near r_0 and a bubble at the origin.

MSC: primary 35B40, 35B45; secondary 35J40

Keywords: Bubbles; Nonlinear Schrödinger equations; Layers

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1. Introduction

We consider *standing waves* for a nonlinear Schrödinger equation in $\mathbb{R}^n$ of the form

$$-i\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^2\Delta\psi - Q(y)\psi + |\psi|^{p-1}\psi, \tag{1.1}$$

where $p > 1$, namely solutions of the form $\psi(t, y) = \exp(i\lambda\varepsilon^{-1}t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that this ψ satisfies (1.1) if and only if u solves the nonlinear elliptic problem

$$\varepsilon^2\Delta u - V(y)u + u^p = 0, \quad u > 0,\ u \in H^1(\mathbb{R}^n), \tag{1.2}$$

where $V(y) = Q(y) + \lambda$. In the rest of this paper, we will assume that V is a smooth function with

$$\inf_{y\in\mathbb{R}^n} V(y) > 0.$$

Considerable attention has been paid in recent years to the problem of construction of standing waves in the so-called *semi-classical limit* of (1.1) $\varepsilon \to 0$. In the pioneering work [19], Floer and Weinstein constructed positive solutions to this problem when $p = 3$ and $n = 1$ with concentration taking place near a given point y_0 with $V'(y_0) = 0$, $V''(y_0) \neq 0$, being exponentially small in ε outside any neighborhood of y_0. More precisely, they established the existence of a solution u_ε such that

$$u_\varepsilon(y) \sim V(y_0)^{\frac{1}{p-1}} w\big(V(y_0)^{\frac{1}{2}}\varepsilon^{-1}(y - y_0)\big),$$

where w is the unique solution of

$$w'' - w + w^p = 0, \quad u > 0, \qquad w'(0) = 0, \qquad w(\pm\infty) = 0. \tag{1.3}$$

This result has been subsequently extended to higher dimensions to the construction of solutions exhibiting high concentration around one or more points of space under various assumptions on the potential and the nonlinearity by many authors. We refer the reader for instance to [1,4,8–17, 20,23,24,28,29,32–35]. In most of the papers, a subcriticality on p is assumed, namely $p < \frac{n+2}{n-2}$.

When $p = \frac{n+2}{n-2}$, there are very few results. Benci and Cerami [6] proved that if $\|\varepsilon^{-2}V(x)\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}$ is small enough, there exists a solution. (That is to say that ε is large.) Concerning the existence of point concentrations (or bubbles) in the case of ε small, Cingolani and Pistoia [10] proved that if $n \geqslant 5$, $V(x) \in L^{\frac{n}{2}}(\mathbb{R}^n)$, then there are no single bubble solutions. Ding and Lin [13] considered (1.2) by adding a subcritical term, namely the following equation

$$\epsilon^2\Delta u - V(x)u + P(x)u^q + K(x)u^{\frac{N+2}{N-2}} = 0, \quad 1 < q < \frac{N+2}{N-2},\ u > 0,\ u \in H^1(\mathbb{R}^n). \tag{1.4}$$

They proved the existence of one solution for ϵ small. Therefore, in general, it seems difficult to construct bubbling solutions for (1.2) with critical exponent.