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Scalar conservation laws with general boundary condition and continuous flux function

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Abstract

We introduce a notion of entropy solution for a scalar conservation law on a bounded domain with nonhomogeneous boundary condition: $u_t + \operatorname{div} \Phi(u) = f$ on $Q = (0,T) \times \Omega$, $u(0,\cdot) = u_0$ on Ω and "$u = a$ on some part of the boundary $(0,T) \times \partial\Omega$." Existence and uniqueness of the entropy solution is established for any $\Phi \in C(\mathbb{R};\mathbb{R}^N)$, $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$, $a \in L^\infty((0,T) \times \partial\Omega)$. In the L^1-setting, a corresponding result is proved for the more general notion of renormalised entropy solution.

Keywords: Conservation law; Nonhomogeneous boundary conditions; Continuous flux; Penalization; L^1-Theory; Renormalized entropy solution

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary if $N > 1$. We consider the following initial boundary value problem for a scalar conservation law:

$$P(u_0, a, f) \quad \begin{cases} \frac{\partial u}{\partial t} + \operatorname{div} \Phi(u) = f & \text{on } Q = (0,T) \times \Omega, \\ u = a & \text{on } \Sigma = (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{on } \Omega, \end{cases}$$

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where $\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous vector field, $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$ and $a \in L^\infty(\Sigma)$.

It is well known that the main difficulty, when dealing with hyperbolic first-order equations, is to make precise the meaning of the boundary condition which may not be assumed pointwise, but has to be read as an entropy condition on the boundary. In the BV-setting, for a smooth flux function Φ and regular data u_0, a, f such an entropy boundary condition has been defined in [1]. However, this condition involved the trace of the BV-solution u and could therefore not be extended to the L^∞-setting. For L^∞-data $u_0, a, f = 0$ and a Lipschitz continuous flux Φ, a new integral formulation of the boundary condition has been given by Otto (cf. [9,12]) who also proved well-posedness of the problem $P(u_0, a, 0)$ in this sense.

For a merely continuous flux function Φ, a different formulation of an entropy solution of $P(u_0, a, f)$ has been proposed in [4] in the particular case of a homogeneous boundary condition, i.e., $a = 0$, and well-posedness has been shown in this setting for arbitrary L^∞-data u_0, f. Following [4] an entropy solution of $P(u_0, 0, f)$ is a function $u \in L^\infty(Q)$ satisfying

$$\int_{\{u>k\}} (u-k)\xi_t + \big(\Phi(u) - \Phi(k)\big) \cdot \nabla\xi + f\xi + \int_{\Omega} (u_0 - k)^+ \xi(0, \cdot) \geqslant 0 \tag{1}$$

for any $(k, \xi) \in \mathbb{R} \times \mathcal{D}([0, T[\times \mathbb{R}^N)$ such that $k \geqslant 0$ and $\xi \geqslant 0$, and for any $(k, \xi) \in \mathbb{R} \times \mathcal{D}([0, T[\times \Omega)$, $\xi \geqslant 0$, and

$$\int_{\{k>u\}} (k-u)\xi_t + \big(\Phi(k) - \Phi(u)\big) \cdot \nabla\xi - f\xi + \int_{\Omega} (k - u_0)^+ \xi(0, x)\, dx \geqslant 0 \tag{2}$$

for any $(k, \xi) \in \mathbb{R} \times \mathcal{D}([0, T[\times \mathbb{R}^N)$ such that $k \leqslant 0$ and $\xi \geqslant 0$, and for any $(k, \xi) \in \mathbb{R} \times \mathcal{D}([0, T[\times \Omega)$, $\xi \geqslant 0$.

In [14], an attempt has been made to extend the definition of entropy solution given by Otto and to prove well-posedness of problem $P(u_0, a, f)$ with a merely continuous flux function Φ. As pointed out in [14], a main difficulty in this case is that BV-a priori estimates seem to be out of reach even when the data u_0, a, f is assumed to be smooth. Due to this lack of strong compactness standard approximation techniques (e.g., by vanishing viscosity) seem to fail. Therefore it seems to be necessary to apply Young measure techniques and to study measure valued entropy solutions of $P(u_0, a, f)$ (cf. [14]).

In this paper we propose a notion of entropy solution of problem $P(u_0, a, f)$ which is a natural generalization of both notions of entropy solutions introduced by Otto and in [4], respectively (cf. Section 2). We prove existence and uniqueness of this entropy solution of problem $P(u_0, a, f)$ for continuous flux Φ and general L^∞-data u_0, a, f, without using Young measure techniques. Instead we apply a very particular approximation technique using penalization which ensures strong compactness in $L^1(Q)$ of the approximate solutions via monotonicity (cf. Sections 3 and 4).

In a quite recent work [13], Vovelle and Porretta have studied problem $P(u_0, a, f)$ in the general L^1-setting. In order to deal with unbounded solutions, they have defined a notion of renormalized entropy solution which generalizes the definition of entropy solutions introduced by Otto in [12] in the L^∞ frame work. They have proved existence and uniqueness of such generalized solution in the case when Φ is locally Lipschitz and the boundary data a verifies the following condition: $\Phi_{\max}(a) \in L^1(\Sigma)$, where $\Phi_{\max}$ is the "maximal effective flux" defined by $\Phi_{\max}(s) = \{\sup |f(t)|,\ t \in [-s^-, s^+]\}$.