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A cooperative system which does not satisfy the limit set dichotomy

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Abstract

The fundamental property of strongly monotone systems, and strongly cooperative systems in particular, is the limit set dichotomy due to Hirsch: if $x < y$, then either $\omega(x) < \omega(y)$, or $\omega(x) = \omega(y)$ and both sets consist of equilibria. We provide here a counterexample showing that this property need not hold for (non-strongly) cooperative systems.

Keywords: Monotone systems; Cooperative systems; Limit set dichotomy

1. Introduction

The field of cooperative, and more generally monotone systems, provides one of the most fruitful areas of theory as well as practical applications—particularly in biology—of dynamical systems. For an excellent introduction, see the textbook by Smith [4] and the recent exposition [3]. One of its central tools is a classical theorem of Hirsch [1,2], the "limit set dichotomy" for strongly monotone (in particular, strongly cooperative)

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systems, see Theorem 1.16 in [3]. The limit set dichotomy states that if $x < y$, then either $\omega(x) < \omega(y)$, or $\omega(x) = \omega(y)$ and both sets consist of equilibria.

According to the recent survey [3], the problem of deciding if there are any cooperative systems for which the dichotomy fails is still open. In [3], Example 1.24, one finds a system which is monotone but not strongly monotone, for which the dichotomy fails. The order in this example is the "ice cream cone" order, and the authors explicitly state that it is unknown whether a polyhedral cone example exists. A cooperative system is one defined by a set of ordinary differential equations $\dot{x} = f(x)$, where $f = (f_1, \ldots, f_n)'$, with the property that $\frac{\partial f_i}{\partial x_j}(x) \geqslant 0$ for all $i \neq j$ and all x. Cooperative systems are monotone with respect to a polyhedral cone, namely the main orthant in $\mathbb{R}^n$. Thus, a counterexample using cooperative systems provides an answer to this open question. We provide such a counterexample here.

To be precise, we construct here two differentiable functions

$$f, g : \mathbb{R} \to \mathbb{R}$$

such that $f(0) = g(0) = 0$, $xf(x) < 0$ and $yg(y) < 0$ for all $x, y \neq 0$, and consider essentially the following system:

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{y} &= g(y), \\ \dot{z} &= x + y. \end{aligned}$$

This system is cooperative. Note that solutions of the x and y equations converge to zero as $t \to \infty$. Moreover, for this system, and for any $\delta > 0$, the following property holds:

There is a solution X with initial condition $(x(0), y(0), z(0))'$ such that $|x(0)| < \delta$ and $|y(0)| < \delta$ so that the omega-limit set $\omega(X)$ is compact and it contains the set

$$\{(0, 0, \zeta) \mid z(0) - \tfrac{1}{2} \leqslant \zeta \leqslant z(0) + \tfrac{1}{2}\}.$$

The limit set dichotomy states would imply that, for any initial conditions

$$(x(0), y(0), z(0))' \quad \text{and} \quad (\hat{x}(0), \hat{y}(0), \hat{z}(0))'$$

for which $x(0) \leqslant \hat{x}(0)$, $y(0) \leqslant \hat{y}(0)$, and $z(0) \leqslant \hat{z}(0)$, and with at least one of the inequalities being strict, the corresponding solutions $X, \hat{X}$ have the property that either $\omega(X) = \omega(\hat{X})$ or $\omega(X) < \omega(\hat{X})$. This last property implies in particular that $\omega(X)$ and $\omega(\hat{X})$ are disjoint. Now, for our example, clearly $\omega(X) \neq \omega(\hat{X})$ as long as $z(0) \neq \hat{z}(0)$ (since the z-components of solutions are translates by $z(0)$ of the solutions with $z(0) = 0$), but the omega-sets intersect as long as the x and y initial conditions are as discussed above, $\hat{z}(0) \leqslant z(0)$, and

$$z(0) - \tfrac{1}{2} < \hat{z}(0) + \tfrac{1}{2},$$