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Asymptotic expansion of the period function II

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Abstract

Let P be a not necessarily bounded polycycle of an analytic vector field on an open set of the plane. Suppose that the singularities which appear after desingularization of the vertices of P are hyperbolic. Consider the function T defined by the return time near P. It is shown that the function T and its derivative T' have asymptotic expansions similar to the series of Dulac but with negative powers.

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1. Introduction

Let E be an analytic ordinary differential equation on an open subset of the plane $\mathbb{R}^2$ and let P be a polycycle of E. Here a *polycycle* is a finite connected union of singularities (*vertices* of P) and integral curves (*sides* of P) of E such that a one-sided return function exists.

Let us consider a transversal section Σ for the polycycle and a local coordinate s along the section whose origin is the intersection of Σ with P. There exist two functions defined on Σ: the *first return map* R and the *return time function* T. The number $T(s)$ is the time required for the integral curve of E through a point of Σ, with a small

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enough coordinate s, to intersect Σ again, for the first time, at $R(s)$. In the case that the polycycle is a boundary of an annulus of periodic orbits, that is when R is the identity map, the return time function is called the *period function*.

We are interested in the qualitative behavior of T. We would like to know the behavior of $T(s)$ when s approaches zero, in particular if T is a nonoscillatory function. Here a function is *nonoscillatory* if the set of its critical points does not have accumulation points.

In [9], we proved that T as well as its derivative T' has an asymptotic expansion in the scale $\{s^{\mu}\}_{\mu\in\mathbb{R}}$ and $\{s^{\delta}\log s\}_{\delta\in\mathbb{R}}$, under the assumption that all the vertices of the polycycle are formally linearizable (after a convenient desingularization). From this, we obtained an almost complete description for the behavior of the return time function associated with an unbounded polycycle in the polynomial case. In this case, the differential equation has a natural extension to the Poincaré sphere, and a polycycle is called *unbounded* if it is not completely contained in the embedded plane $\mathbb{R}^2$, that is, when one of the vertices is contained in the equator.

In this work we adopt the same notation as used in [9]. That is, $\widetilde{X}$ denotes the vector field obtained after the desingularization of the vertices of P and $\widetilde{P}$ is the corresponding polycycle with desingularized vertices (the Jacobian matrix of $\widetilde{X}$ at a vertex is not nilpotent) [3,10]. We say that (E,P) is *hyperbolic after desingularization* if the vertices of $\widetilde{P}$ are hyperbolic, that is, if none of the eigenvalues of the Jacobian matrix of $\widetilde{X}$ at a vertex are zero. Since P has a one-sided return function, these eigenvalues have opposite signs. We prove, in this case, that both T and its derivative T' have asymptotic expansions in the scale $\{s^{\mu}(\log s)^n\}_{\mu\in\mathbb{R},\ n\in\mathbb{N}\cup\{0\}}$. More precisely, our main result is:

Theorem 1. *If (E,P) is hyperbolic after desingularization, then the return time function T of P has an asymptotic expansion $\widehat{T}$ given by*

$$\widehat{T}(s)=\sum_{k\in\mathbb{N}} s^{\mu_k}P_k(\log s),\qquad s>0, \tag{1}$$

where the sequence $\{\mu_k\}$ of real numbers is strictly increasing and unbounded and P_k is a polynomial with real coefficients. Moreover, the asymptotic expansion of the derivative T' of T is the formal derivative of $\widehat{T}$, denoted $\widehat{T}'$.

From this we conclude that the return time function has a nonoscillatory behavior whenever the asymptotic expansion of T' is different from zero. This is the case when at least one of the vertices of P is at a finite distance. It follows that the return time function is monotone. More precisely, we have

Corollary 1. *If (E,P) is hyperbolic after desingularization and P has at least one vertex at a finite distance, then the first term in series (1) has the form $s^{\mu}(\log s)^{\beta}$, with $\mu\leqslant 0$, $\beta=0,1$ and $(\mu,\beta)\neq(0,0)$. In particular, the function T is monotone on a neighborhood of P.*