J. Math. Anal. Appl. 447 (2017) 1–16

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Discrete spectra of compactly perturbed bounded operators

Michael Gil'

Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel

A R T I C L E I N F O

Article history:
Received 16 December 2015
Available online 5 October 2016
Submitted by L. Fialkow

Keywords:
Schatten–von Neumann operators
Perturbations
Quasinormed ideals

A B S T R A C T

Let C be a bounded operator on a Banach space or on a Hilbert space, and consider the operator $A = C + K$, where K is a compact operator. We are interested in the discrete spectrum of A in domains free of the spectrum of C. In the first part of the paper we deal with Hilbert space operators assuming that K is a Schatten–von Neumann operator. Besides, the bounds for the absolute values and imaginary parts of the eigenvalues of A are obtained in terms of the Schatten–von Neumann norm of K and norm of the resolvent of C. In addition, we estimate the counting functions of the numbers of the eigenvalues of A in various domains. In the second part we particularly generalize our results to so called p-quasi-normed ideals of compact operators in a Banach space. Our main tool is a combined usage of the regularized determinant of the operator $zK(I - zC)^{-1}$ ($z \in \mathbb{C}$), where I is the unit operator, and recent norm estimates for resolvents. Applications of our results to the non-selfadjoint Jacobi operator are also discussed.

1. Introduction

Denote by $\mathcal{H}$ a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$, and by $\mathcal{X}$ a separable Banach space with a norm $\|.\|_{\mathcal{X}}$. By $[\mathcal{X}]$ we denote the set of all bounded linear operators in $\mathcal{X}$. For a linear operator A, A^* is the adjoint operator; $\sigma(A)$ is the spectrum; $\sigma_{ess}(A)$ denotes the essential spectrum. $r_s(A)$ is the spectral radius; $SN_p = SN_p(\mathcal{H})$ $(1 \le p < \infty)$ is the Schatten-von Neumann ideal of operators K_0 in $\mathcal{H}$ with the finite norm

$$N_p(K_0) := [\text{trace } (K_0^* K_0)^{p/2}]^{1/p}.$$

The study of the distribution of eigenvalues of operators on $\mathcal{X}$ is a classical and well-developed subject (see e.g., the monographs [15,17]). While the distribution of eigenvalues of compact operators is well-studied, the noncompact ones are investigated considerably less than compact operators. Here we should point the papers [2–4,12,13] and references given therein. The papers [2,3,12] deal with distribution of eigenvalues

E-mail address: gilmi@bezeqint.net.

http://dx.doi.org/10.1016/j.jmaa.2016.10.005
0022-247X/

of non-compact operators on $\mathcal{H}$ of the form $A = C + K$, where $C \in [\mathcal{H}]$ and $K \in SN_p$. In particular, perturbation results and bounds for the counting functions of the eigenvalues of A have been established. In [4] the results from [2,3] were extended to operators in a Banach space. The role of the singular numbers is played by the approximation numbers. In the paper [13] the approach from [4] has been generalized to the case when K belongs to so called p-quasinormed ideals of compact operators in $\mathcal{X}$.

In the present paper we combine a method based on complex analysis, developed in the papers [2–4] with estimates on norms of resolvents of certain classes of operators, obtained in [7], in order to derive some new and concrete estimates on eigenvalues. In addition, the present paper introduces the idea of using the spectral mapping theorem to obtain the results about the number of eigenvalues in different domains in the complex plane from results regarding the number of eigenvalues outside a disk, in particular allowing to obtain bounds on imaginary parts of eigenvalues.

A few words about the contents. The paper consists of 9 sections.

In Sections 2 and 3 we derive a bound for the counting function of the eigenvalues of $C + K$ in the set $\{z \in \mathbb{C} : |z| > r_s(C)\}$. The results of these sections are close to the ones from [3, Section 3] but in a more convenient form for us. In Section 4 we detail the results from Section 2 for some non-selfadjoint operators. In Section 5 by virtue of the results from the previous sections we derive bounds for the counting functions in various domains of the complex plane.

In Sections 6 and 7 we estimate the absolute values and imaginary parts of the eigenvalues of $C + K$, respectively. In Section 8, we apply our above results to estimate the eigenvalues of non-selfadjoint Jacobi operators. In Section 9, we particularly generalize our results from Section 2 to the so called p-quasi-normed ideal of compact operators in a Banach space.

2. Operators in a Hilbert space

Assume that

$$C \in [\mathcal{H}] \text{ and } K \in SN_p \text{ for an integer } p \geq 1. \tag{2.1}$$

Denote by $\nu(C + K, r)$ the number of the eigenvalues of $C + K$ with their (algebraic) multiplicities in the set $\{z \in \mathbb{C} : |z| \geq r\}$ $(r > r_s(C))$.

Note that by Weyl's theorem on the preservation of the essential spectrum under compact perturbations, we have $\sigma_{ess}(C) = \sigma_{ess}(C + K)$ (see [14, Theorem 5.35, p. 244]). We are interested in the discrete spectrum of $C + K$. To this end put

$$\zeta_p = (p-1)/p \ \ (p \neq 1, p \neq 3); \ \ \zeta_1 = \zeta_3 = 1.$$

Theorem 2.1. *Let condition (2.1) hold. Then for all $r > r_s(C)$ and $R \in (r_s(C), r)$ one has*

$$\nu(C + K, r) \leq \frac{\zeta_p}{\ln (r/R)} \left(N_p(K) \sup_{|z|=R} \|(C - zI)^{-1}\| \right)^p .$$

This theorem is proved in the next section. In particular, with $R = r/e > r_s(C)$ we have

$$\nu(C + K, r) \leq \zeta_p \left(N_p(K) \sup_{|z|=r/e} \|(C - zI)^{-1}\| \right)^p .$$

From the latter inequality it follows that $C + K$ does not have eigenvalues in the set $\{z \in \mathbb{C} : |z| \geq r_0\}$ $(r_0 > r_s(C)e)$, provided $N_p(K) \sup_{|z|=r_0/e} \|(zI - C)^{-1}\| < 1$. This conclusion characterizes the sharpness