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Conical square functions associated with Bessel, Laguerre and Schrödinger operators in UMD Banach spaces ☆

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ABSTRACT

In this paper we consider conical square functions in the Bessel, Laguerre and Schrödinger settings where the functions take values in UMD Banach spaces. Following a recent paper of Hytönen, van Neerven and Portal [36], in order to define our conical square functions, we use γ-radonifying operators. We obtain new equivalent norms in the Lebesgue–Bochner spaces $L^p((0,\infty),\mathbb{B})$ and $L^p(\mathbb{R}^n,\mathbb{B})$, $1<p<\infty$, in terms of our square functions, provided that $\mathbb{B}$ is a UMD Banach space. Our results can be seen as Banach valued versions of known scalar results for square functions.

1. Introduction

In this paper we obtain equivalent norms in the Lebesgue–Bochner space $L^p(\mathbb{R}^n,\mathbb{B})$, $1<p<\infty$, where $\mathbb{B}$ is a UMD Banach space, in terms of conical square functions defined via fractional derivatives of Poisson semigroups associated with Bessel, Laguerre and Schrödinger operators. According to the ideas developed by Hytönen, van Neerven and Portal [36] we use appropriate tent spaces using γ-radonifying operators (or, in other words, methods of stochastic analysis in a Banach valued setting).

We denote by $P_t(z)$, the classical Poisson kernel in $\mathbb{R}^n$, that is,

$$P_t(z) = c_n \frac{t}{(|z|^2+t^2)^{(n+1)/2}}, \quad t>0 \text{ and } z\in\mathbb{R}^n,$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$.

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Segovia and Wheeden [50] introduced fractional derivatives as follows. Suppose that $\beta > 0$ and $m \in \mathbb{N}$ is such that $m - 1 \leq \beta < m$. If $F : \Omega \times (0, \infty) \longrightarrow \mathbb{C}$ is a reasonable nice function, where $\Omega \subset \mathbb{R}^n$, the β-th derivative with respect to t of F is defined by

$$\partial_t^\beta F(x,t) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m F(x, t+s) s^{m-\beta-1} ds \quad t > 0 \text{ and } x \in \Omega.$$

In [50] this fractional derivative was used to get characterizations of classical Sobolev spaces.

As in [58] we define the β-conical square function S_β by

$$S_\beta(f)(x) = \left(\int_{\Gamma(x)} \left| t^\beta \partial_t^\beta P_t(f)(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where $P_t(f)$ denotes the Poisson integral of f, that is,

$$P_t(f)(x) = \int_{\mathbb{R}^n} P_t(x-y) f(y) dy, \quad x \in \mathbb{R}^n, t > 0, \tag{1}$$

and, for every $x \in \mathbb{R}^n$, $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |x-y| < t\}$. According to [58, Theorems 5.3 and 5.4] the square function S_β defines an equivalent norm in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Theorem A. *Let $1 < p < \infty$ and $\beta > 0$. Then, there exists $C > 0$ such that*

$$\frac{1}{C} \|f\|_{L^p(\mathbb{R}^n)} \leq \|S_\beta(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n). \tag{2}$$

The equivalence in Theorem A for $\beta \in \mathbb{N}$ can also be seen in [36,42,52].

Coifman, Meyer and Stein [20] introduced a family of spaces called tent spaces. These tent spaces are well adapted to certain questions related to harmonic analysis. Suppose that $1 \leq p, q < \infty$. The tent space $T_p^q(\mathbb{R}^n)$ consists of all those measurable functions g on $\mathbb{R}^n \times (0,\infty)$ such that $A_q(g) \in L^p(\mathbb{R}^n)$, where

$$A_q(g)(x) = \left(\int_{\Gamma(x)} |g(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

The norm $\| \cdot \|_{T_p^q(\mathbb{R}^n)}$ in $T_p^q(\mathbb{R}^n)$ is defined by $\|g\|_{T_p^q(\mathbb{R}^n)} = \|A_q(g)\|_{L^p(\mathbb{R}^n)}$, $g \in T_p^q(\mathbb{R}^n)$.

More recently Harboure, Torrea and Viviani [32] have simplified some proofs of properties in [20] by using vector valued harmonic analysis techniques. Note that the result in Theorem A can be rewritten in terms of tent spaces as follows. If $1 < p < \infty$ and $\beta > 0$, then, for every $f \in L^p(\mathbb{R}^n)$, $t^\beta \partial_t^\beta P_t(f) \in T_p^2(\mathbb{R}^n)$ and

$$\frac{1}{C} \|f\|_{L^p(\mathbb{R}^n)} \leq \|t^\beta \partial_t^\beta P_t(f)\|_{T_p^2(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)},$$

where $C > 0$ does not depend on f.

Assume that $\mathbb{B}$ is a Banach space. In order to show a version of Theorem A for the Lebesgue–Bochner space $L^p(\mathbb{R}^n, \mathbb{B})$, the most natural definition of the β-conical square function $S_{\beta,\mathbb{B}}$ is the following