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Unique conditional expectations for abelian C^*-inclusions

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ABSTRACT

Let $\mathcal{D} \subseteq \mathcal{A}$ be an inclusion of unital abelian C^*-algebras. In this note we characterize (in topological terms) when there is a unique conditional expectation $E : \mathcal{A} \to \mathcal{D}$, at least when $\mathcal{A}$ is separable. We also provide the first example of an inclusion with a unique conditional expectation, but multiple pseudo-expectations (in the sense of Pitts).

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1. Introduction

A C^**-inclusion* is a pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ is a unital C^*-algebra and $\mathcal{D} \subseteq \mathcal{C}$ is a unital C^*-subalgebra (with the same unit). Ideally, the study of C^*-inclusions can lead to a better understanding of the containing algebra $\mathcal{C}$, via "coordinatization" with respect to the included algebra $\mathcal{D}$. Seminal works in this direction are Kumjian's paper on C^**-diagonals* [5] and Renault's paper on *Cartan subalgebras in* C^**-algebras* [10]. These, in turn, were motivated by Feldman and Moore's study of Cartan subalgebras in von Neumann algebras [3]. In particular, in both Kumjian and Renault's settings, there is a unique conditional expectation from $\mathcal{C}$ onto $\mathcal{D}$, which is faithful.

A *conditional expectation* from $\mathcal{C}$ onto $\mathcal{D}$ is a contractive linear projection $E : \mathcal{C} \to \mathcal{D}$. Conditional expectations enjoy many nice properties—they are idempotent, unital completely positive (ucp) maps, and are $\mathcal{D}$-bimodular (i.e., $E(d_1 x d_2) = d_1 E(x) d_2$ for all $x \in \mathcal{C}$, $d_1, d_2 \in \mathcal{D}$) [15]. A conditional expectation is *faithful* if $E(x^*x) = 0$ implies $x = 0$. Unfortunately, it can easily happen that a C^*-inclusion admits no conditional expectations. For example, if $\mathcal{C}$ is injective (in the category OpSys of operator systems and ucp maps), but $\mathcal{D}$ is not, then $(\mathcal{C}, \mathcal{D})$ admits no conditional expectations. In particular, $(L^\infty[0,1], C[0,1])$ admits no conditional expectations.

In [8], Pitts introduced *pseudo-expectations* as a substitute for possibly non-existent conditional expectations. A pseudo-expectation for the C^*-inclusion $(\mathcal{C}, \mathcal{D})$ is a ucp map $\theta : \mathcal{C} \to I(\mathcal{D})$ such that

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$\theta|_{\mathcal{D}} = \mathrm{id}$. Here $I(\mathcal{D})$ is Hamana's *injective envelope* of $\mathcal{D}$, the (essentially) unique minimal injective in OpSys containing $\mathcal{D}$ [4]. In fact, $I(\mathcal{D})$ is a unital C^*-algebra containing $\mathcal{D}$ as a unital C^*-subalgebra. By injectivity, pseudo-expectations always exist for any C^*-inclusion. Every conditional expectation is a pseudo-expectation. Like conditional expectations, pseudo-expectations are contractive and $\mathcal{D}$-bimodular. Unlike conditional expectations, pseudo-expectations are not idempotent. In fact, the composition $\theta \circ \theta$ is generally undefined, since $I(\mathcal{D})$ rarely sits inside $\mathcal{C}$.

In [9], Pitts and the author made a systematic study of pseudo-expectations, with an emphasis on determining which C^*-inclusions admit a unique pseudo-expectation. For *abelian C^*-inclusions* (i.e., C^*-inclusions $(\mathcal{A}, \mathcal{D})$ with $\mathcal{A}$ abelian), we found an elegant necessary and sufficient condition for there to exist a unique pseudo-expectation. In order to state our result, we remind the reader that by the Gelfand–Naimark Theorem, $\mathcal{A} \cong C(Y)$ (the continuous complex-valued functions on a compact Hausdorff space Y), $\mathcal{D} \cong C(X)$, and the inclusion map $\iota : C(X) \to C(Y)$ is determined by a continuous surjection $j : Y \to X$ via the formula $\iota(f) = f \circ j$, $f \in C(X)$.

Theorem 1.1 *([9], Cor. 3.21). Let $j : Y \to X$ be a continuous surjection of compact Hausdorff spaces. Then the corresponding abelian C^*-inclusion $(C(Y), C(X))$ admits a unique pseudo-expectation if and only if there exists a unique minimal closed set $K \subseteq Y$ such that $j|_K : K \to X$ is a surjection.*

The main purpose of this note is to find an analogous topological characterization of when $(C(Y), C(X))$ admits a unique conditional expectation. We are able to do so under the restriction that Y is a compact metric space (equivalently, that $C(Y)$ is separable).

Theorem 1.2. *Let $j : Y \to X$ be a continuous surjection of compact metric spaces. Then the corresponding abelian C^*-inclusion $(C(Y), C(X))$ admits a unique conditional expectation if and only if there exists a unique G_δ set $A \subseteq Y$ such that $j|_A : A \to X$ is an open surjection. In that case, $A \subseteq Y$ is closed and $j|_A : A \to X$ is a homeomorphism.*

Even though this result appears classical, it is new, to the author's knowledge. Certainly the proof is not classical, since it relies on fairly recent results about *regular averaging operators* and *exact Milutin maps* [11,1].

We also produce an abelian C^*-inclusion with a unique conditional expectation but multiple pseudo-expectations (Example 4.4). This answers Question 3 in Section 7.1 of [9] affirmatively.

2. Preliminaries

As mentioned in the introduction, the proof of Theorem 1.2 consists mainly of appealing to known results from the theory of regular averaging operators and exact Milutin maps. We collect the relevant results here for the reader's convenience, restating them in the form best suited to our purposes. We only include proofs when our restatement appears stronger than the original.

For a compact Hausdorff space Y (or for a Polish space Y), $C_b(Y)$ will denote the absolutely-bounded continuous complex-valued functions on Y equipped with the supremum norm, $\mathcal{M}(Y)_+ \subseteq C_b(Y)^*$ will denote the finite non-negative regular Borel measures on Y, equipped with the weak* topology, and $\mathcal{P}(Y) \subseteq \mathcal{M}(Y)_+$ will denote the regular Borel probability measures on Y. For $\mu \in \mathcal{M}(Y)_+$, we define

$$\mathrm{supp}(\mu) = \{y \in Y : \mu(U) > 0 \text{ for every open set } U \subseteq Y \text{ containing } y\}.$$

It is easy to see that $\mathrm{supp}(\mu) \subseteq Y$ is closed. For $\mu \in \mathcal{P}(Y)$, we have that

$$\mathrm{supp}(\mu) = \bigcap\{F : F \subseteq Y \text{ is a closed set such that } \mu(F) = 1\}.$$