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Characterization of the potential smoothness of one-dimensional Dirac operator subject to general boundary conditions and its Riesz basis property

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ABSTRACT

The one-dimensional Dirac operator with periodic potential $V = \begin{pmatrix} 0 & \mathscr{P}(x) \\ \mathscr{Q}(x) & 0 \end{pmatrix}$, where $\mathscr{P}, \mathscr{Q} \in L^2([0,\pi])$ subject to periodic, antiperiodic or a general strictly regular boundary condition (bc), has discrete spectrums. It is known that, for large enough $|n|$ in the disk centered at n of radius $1/2$, the operator has exactly two (periodic if n is even or antiperiodic if n is odd) eigenvalues λ_n^+ and λ_n^- (counted according to multiplicity) and one eigenvalue μ_n^{bc} corresponding to the boundary condition (bc). We prove that the smoothness of the potential could be characterized by the decay rate of the sequence $|\delta_n^{bc}|+|\gamma_n|$, where $\delta_n^{bc} = \mu_n^{bc} - \lambda_n^+$ and $\gamma_n = \lambda_n^+ - \lambda_n^-$. Furthermore, it is shown that the Dirac operator with periodic or antiperiodic boundary condition has the Riesz basis property if and only if $\sup\limits_{\gamma_n \neq 0} \frac{|\delta_n^{bc}|}{|\gamma_n|}$ is finite.

1. Introduction

We consider the one-dimensional Dirac operator

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} 0 & \mathscr{P}(x) \\ \mathscr{Q}(x) & 0 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1}$$

where $\mathscr{P}, \mathscr{Q} \in L^2([0,\pi])$, with periodic, antiperiodic and Dirichlet boundary conditions. We also consider a general boundary condition (bc) given by

$$\begin{aligned} y_1(0) + by_1(\pi) + ay_2(0) &= 0, \\ dy_1(\pi) + cy_2(0) + y_2(\pi) &= 0, \end{aligned} \tag{2}$$

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where a, b, c, d are complex numbers subject to the restrictions

$$b + c = 0, \quad ad = 1 - b^2 \tag{3}$$

and

$$ad \neq 0. \tag{4}$$

It is well known that if $\mathscr{P}, \mathscr{Q} \in L^2([0, \pi])$, $\mathscr{P} = \overline{\mathscr{Q}}$ and we extend $\mathscr{P}$ and $\mathscr{Q}$ as π-periodic functions on $\mathbb{R}$, then the operator is self-adjoint and has a band-gap structured spectrum of the form

$$Sp(L) = \bigcup_{n=-\infty}^{+\infty} [\lambda_{n-1}^+, \lambda_n^-],$$

where

$$\cdots \leq \lambda_{n-1}^+ < \lambda_n^- \leq \lambda_n^+ < \lambda_{n+1}^- \cdots .$$

In addition, Floquet theory shows that the endpoints $\lambda_n^\pm$ of these spectral gaps are eigenvalues of the operator (1) subject to periodic boundary conditions or antiperiodic boundary conditions. Furthermore, the spectrum is discrete for each of the above boundary conditions. Also, for $n \in \mathbb{Z}$ with large enough $|n|$ the disk with center n and radius $1/2$ contains two eigenvalues (counted with multiplicity) λ_n^+ and λ_n^- of periodic (if n is even) or antiperiodic (if n is odd) boundary conditions and as well one eigenvalue μ_n^{Dir} of Dirichlet boundary condition. There is also one eigenvalue μ_n^{bc} of the general boundary condition (bc) given above (which will be proven in the first section).

There is a very close relationship between the smoothness of the potential and the rate of decay of the deviations $|\lambda_n^+ - \lambda_n^-|$ and $|\mu_n^{Dir} - \lambda_n^+|$. The story of the discovery of this relation was initiated by H. Hochstadt [14,15] who considered the (self-adjoint) Hill's operator and proved that the decay rate of the spectral gap $\gamma_n = |\lambda_n^+ - \lambda_n^-|$ is $O(1/n^{m-1})$ if the potential has m continuous derivatives. Furthermore, he showed that every finite-zone potential (i.e., $\gamma_n = 0$ for all but finitely many n) is a C^∞-function. Afterwards, some authors [18–20] studied on this relation and showed that if γ_n is $O(1/n^k)$ for any $k \in \mathbb{Z}^+$, then the potential is infinitely differentiable. Furthermore, Trubowitz [25] showed that the potential is analytic if and only if γ_n decays exponentially fast. In the non-self-adjoint case, the potential smoothness still determines the decay rate of γ_n. However, the decay rate of γ_n does not determine the potential smoothness as Gasymov showed [10]. In this case, Tkachenko [23,21,24] gave the idea to consider γ_n together with the deviation $\delta_n^{Dir} = \mu_n^{Dir} - \lambda_n^+$ and obtained characterizations of C^∞-smoothness and analyticity of the potential with these deviations γ_n and δ_n^{Dir}. In addition to these developments, Sansuc and Tkachenko [22] proved that the potential is in the Sobolev space H^m, $m \in \mathbb{N}$, if and only if γ_n and δ_n^{Dir} satisfy

$$\sum (|\gamma_n|^2 + |\delta_n^{Dir}|^2)(1 + n^{2m}) < \infty.$$

The results mentioned above have been obtained by using Inverse Spectral Theory.

Grébert, Kappeler, Djakov and Mityagin studied the relationship between the potential smoothness and the decay rate of spectral gaps for Dirac operators (see [13,12,4]).

We recall that a characterization of smoothness of a function can be given by weights $\Omega = \Omega(n)_{n\in\mathbb{Z}}$, where the corresponding weighted Sobolev space is

$$H(\Omega) = \{v(x) = \sum_{k\in\mathbb{Z}} v_k e^{2ikx} : \quad \sum_{k\in\mathbb{Z}} |v_k|^2 (\Omega(k))^2 < \infty\}$$