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Infinite-dimensional features of matrices and pseudospectra

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Given a Hilbert space operator T, the level sets of function $\Psi_T(z) = \|(T-z)^{-1}\|^{-1}$ determine the so-called pseudospectra of T. We set Ψ_T to be zero on the spectrum of T. After giving some elementary properties of Ψ_T (which, as it seems, were not noticed before), we apply them to the study of the approximation. We prove that for any operator T, there is a sequence $\{T_n\}$ of finite matrices such that $\Psi_{T_n}(z)$ tends to $\Psi_T(z)$ uniformly on $\mathbb{C}$. In this proof, quasitriangular operators play a special role. This is merely an existence result, we do not give a concrete construction of this sequence of matrices.

One of our main points is to show how to use infinite-dimensional operator models in order to produce examples and counterexamples in the set of finite matrices of large size. In particular, we get a result, which means, in a sense, that the pseudospectrum of a nilpotent matrix can be anything one can imagine. We also study the norms of the multipliers in the context of Cowen–Douglas class operators. We use these results to show that, to the opposite to the function Ψ_S, the function $\|\sqrt{S-z}\,\|$ for certain finite matrices S may oscillate arbitrarily fast even far away from the spectrum.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$, equipped with the supremum norm. Given an operator $T \in \mathcal{B}(\mathcal{H})$, put

$$\Psi_T(z) = \begin{cases} 0 & \text{if } z \in \sigma(T); \\ \|(T-z)^{-1}\|^{-1} & \text{if } z \notin \sigma(T). \end{cases}$$

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This function is closely related with so-called ε-pseudospectra of T, defined by

$$\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \Psi_T(z) < \varepsilon\}$$

(here $\varepsilon > 0$). It is well known that

$$\sigma_\varepsilon(T) = \bigcup_{\|A\|<\varepsilon} \sigma(T + A),$$

see, for instance [15,27,44]. While the ε-pseudospectrum of a normal operator in a Hilbert space coincides with the ε-neighborhood of the spectrum, the situation is more involved for non-normal operators. It is well-known that the spectral properties of a nonnormal operator (or matrix) not only depend on its spectrum, but are also influenced by the resolvent growth. The pseudospectra are a good language to describe this growth, and their importance has been widely recognized in the recent years. Their applications include the finite section method for Toeplitz matrices, growth bounds for semigroups, numerics for differential operators, matrix iterations, linear models for turbulence, etc. We refer to the book [50] by Trefethen and Embree and to the Trefethen's review [48] for comprehensive accounts. Much effort has been devoted to the calculation of pseudospectra of matrices [49].

By a *filtration* on $\mathcal{H}$, we mean a sequence $\{P_n\}$ of finite rank orthogonal projections such that $\operatorname{Ran} P_n \subseteq \operatorname{Ran} P_{n+1}$ and $\cup_n \operatorname{Ran} P_n$ is dense in $\mathcal{H}$. The corresponding sequence of finite dimensional operators $T_n = P_n T_{|\operatorname{Ran} P_n}$ will be referred to as *finite sections of* T.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *quasitriangular* if there is a filtration $\{P_n\}$ such that $\lim_{n\to\infty} \|(I - P_n)TP_n\| = 0$. If there is a filtration $\{P_n\}$ such that both $\lim_{n\to\infty} \|(I - P_n)TP_n\| = 0$ and $\lim_{n\to\infty} \|P_nT(I - P_n)\| = 0$, then T is said to be quasidiagonal. In these cases, we will refer to $\{P_n\}$ as to a *filtration, corresponding to a quasitriangular (quasidiagonal) operator* T.

It is well known that spectra do not necessarily behave well under limiting procedures, even for a sequence of bounded operators on some Hilbert space $\mathcal{H}$ converging in operator norm. For example, consider the bilateral weighted shift on $\ell_2(\mathbb{Z})$, defined by $T(s)e_j = e_{j+1}$ for $j \neq 0$ and $T(s)e_0 = se_1$ (here $\{e_j : j \in \mathbb{Z}\}$ is the standard basis of $\ell_2(\mathbb{Z})$). Then the spectrum of $T(s)$ equals to the unit circle for any nonzero s, while the spectrum of $T(0)$ is the whole closed unit disc, and there is no convergence of spectra as $s \to 0$. For the case of pseudospectra, the situation is better. It was noticed by many authors that, to the opposite to usual spectra, pseudospectra supply a vast quantitative information on the behavior of powers of operators, the semigroups they generate, etc. Our work also gives some results in this direction.

Our main results are as follows. In Section 2, we prove several elementary estimates and properties for the function $\Psi_T(z)$. In particular, we show that it is locally semiconvex (see the definition below). The list of these properties certainly can be extended. However, the question of describing all functions on $\mathbb{C}$ representable as $\Psi_T(z)$ for a Hilbert (or Banach) space operator T seems to be open and might be interesting. We use the results of Section 2 in the next sections. We believe that these results may also be important for algorithms of numerical calculation of pseudospectra.

Section 3 is devoted to general convergence results for pseudospectra and for the function $\Psi_T(z)$. One of our starting points was the result by N. Brown, which says that if T is quasidiagonal operator and $\{P_n\}$ is a corresponding filtration, then for any ε, the ε-pseudospectra of T_n tend to the ε-pseudospectrum of T, see [13], Theorem 3.5 (1). We observe that a similar assertion holds also for quasitriangular operators. We prove that for any quasitriangular operator T and the corresponding filtration $\{P_n\}$, the functions Ψ_{T_n} tend uniformly to Ψ_T on the whole complex plane. This permits us to show that for an arbitrary operator T, there is a sequence of matrices S_n such that Ψ_{S_n} tend uniformly to Ψ_T on $\mathbb{C}$. Here we use the theorem by Apostol, Foiaş and Voiculescu, which characterizes quasitriangular operators in terms of semi-Fredholmness.

In Section 4, we use the above convergence results to prove that, in a sense, the function $\Psi_T(z)$, corresponding to a nilpotent matrix T, can have any imaginable shape. In this proof, we apply our approximation