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An exact estimate result for a semilinear equation with critical exponent and prescribed singularity

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ABSTRACT

We consider the singular boundary value problem

$$\begin{cases} -\Delta u - \lambda V(x)u = h(x)u^{-\gamma} + \mu u^{2^*-1} & in\ \Omega, \\ u(x) > 0 & in\ \Omega, \\ u(x) = 0 & on\ \partial\Omega, \end{cases}$$

where $0 < \lambda < \bar{\lambda} = (\frac{N-2}{2})^2$, $0 < \gamma < 1$, $2^* = \frac{2N}{N-2}$, $h(x)$ is a given function and $V(x)$ has prescribed finitely many singular points. Our goal in this paper is to establish some existence and multiplicity results for above problem when $\mu \in (0, \mu^*)$ for some $\mu^* > 0$ and obtain exact estimate for extremal value $\mu^* = \mu^*(\Omega, \gamma, 2^*, h(x)) > 0$ for above problem.

1. Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} -\Delta u - \lambda V(x)u = h(x)u^{-\gamma} + \mu u^{2^*-1} & in\ \Omega, \\ u(x) > 0 & in\ \Omega, \\ u(x) = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, Δ is the Laplace operator, $\mu > 0$ is a parameter, $0 < \lambda < \bar{\lambda} = \left(\frac{N-2}{2}\right)^2$, and $\bar{\lambda}$ is the best Hardy constant. $0 < \gamma < 1$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Throughout our paper, we assume that $h \in C(\overline{\Omega})$ and $h(x) > 0$.

Let $H_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$. Problem (1.1) is related to the well-known Hardy inequality [23]:

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$$\int_\Omega \frac{|u|^2}{|x-a|^2}dx \le \frac{1}{\bar{\lambda}}\int_\Omega |\nabla u|^2 dx, \quad \forall\, a\in\mathbb{R}^N, \quad \forall\, u\in C_0^\infty(\Omega).$$

The linear weight $V(x)$ has finitely many singular points and we will impose the following assumptions on $V(x)$:
(V_1) there exist $a_1,\ a_2,\ \cdots, a_k\in\Omega$ such that $V(x)\in L^\infty_{loc}(\Omega\backslash\{a_1,a_2,\cdots,a_k\})$ and

$$\lim_{x\to a_i} V(x)|x-a_i|^2 = 1, \quad i\in\{1,2,\cdots,k\}.$$

Moreover, there exists $\delta>0$, $0<\alpha,\beta<2\sqrt{\bar{\lambda}-\lambda}$ such that $|a_i-a_j|\ge 4\delta$ for $i\ne j$, $B(a_i,4\delta)\subset\Omega$, and

$$1-|x-a_i|^\alpha \le |x-a_i|^2V(x)\le 1-|x-a_i|^\beta, \quad i=1,2,\cdots,k,$$

for every $x\in B(a_i,2\delta)\backslash\{a_1,a_2,\cdots,a_k\}$;
(V_2) there exists a constant C with $0<C<1$ such that

$$\int_\Omega \lambda V(x)u^2dx \le C\int_\Omega |\nabla u|^2dx, \quad u\in H_0^1(\Omega).$$

In Quantum Chemistry, multi-singular potentials stem from molecular systems consisting of k nuclei of unit charge located at a finite number of points $a_1,\ a_2,\ \cdots,\ a_k$ and of k electrons. Coulomb multi-singular potentials appear in the interactions between the fixed nuclei and the electrons, see Catto et al. [9], Lions [29], Felli and Terracini [17], Cao and Han [7], Hsu [25]. In addition, problem (1.1) can also act as a model for many problems coming from astrophysics, cosmology, and differential geometry [4,10,24,26,28].

In recent years, much attention has been paid to elliptic problems involving singular nonlinearity (1.1) [2,14,19,20,31,32]. In the case of $\lambda=0$, by defining $T_\mu=\inf\{\mu>0:$ (1.1) *has no weak solution*$\}$, Giacomoni et al. [19] showed the existence of two positive solutions of problem (1.1) with $h(x)=1$ for every $\mu\in(0,T_\mu)$: a saddle point for the energy functional corresponding to (1.1) and a local minimizer. Furthermore, Sun and Wu [32] used variational arguments based on Nehari's tool to obtain the dependence of T_μ on $\Omega,\ 2^*,\ \gamma$ and $h(x)$. Indeed, the authors gave a complete description of a constraint set associated to the energy functional. Recently, Cong and Han [16] proved that problem (1.1) with $\mu=1$ and $\gamma>1$ admits a solution if and only if there exists $u_0\in H_0^1(\Omega)$ such that $\int_\Omega h(x)u_0^{1-\gamma}dx<\infty$. In the case of $\lambda\ne 0$, we should point out that multiplicity results for positive solutions of problem (1.1) with $V(x)=|x|^{-2}$ have been obtained in Chen and Rocha [14].

On the other hand, problem (1.1) with $h(x)\equiv 0$ has also been studied by some authors [12,15,13,17,18,22,25–28]. We would like to mention the results of [12,15,13,18,30], which motivated us to discuss (1.1). Smets [30] showed that blowing up positive Palais–Smale sequences of the functional corresponding to (1.1) with $V(x)=|x|^{-2}$ and $\mu=1$ can bear exactly two kind of bubbles. When $\lambda V(x)$ is replaced by $\lambda_1|x|^{-2}+\lambda_2$, Ferrero and Gazzola [18] have proved that there is one nontrivial solution of (1.1) with $\mu=1$ and $N\ge 4$ and any $\lambda_1>0$ but $\lambda_2\notin\sigma_{\lambda_1}$ (σ_{λ_1} is the spectrum of the operator $-\Delta-\lambda_1|x|^{-2}$ in $H_0^1(\Omega)$). Very recently, we note that Chen and Chen [13] dealt with the existence and multiplicity of positive solutions for the following semilinear equation with critical exponent and prescribed singularity

$$-\Delta u-\mu V(x)u=|u|^{2^*-2}u+\theta h(x), \quad \forall u\in H_0^1(\Omega), \tag{1.2}$$

where $V(x)$ satisfies the conditions (V_1) and (V_2). The starting point of [13] is the works due to Chen [12] and Chen et al. [15]. Chen [12] had showed that the problem (1.1) has at least k positive solutions in $H_0^1(\Omega)$ under the hypotheses of Theorem 1.1 in Chen and Chen [13]. Moreover, by completing and refining the analysis performed in Chen [12], Chen–Chen [13] proved $2k$ positive weak solutions for problem (1.2).