J. Math. Anal. Appl. 447 (2017) 154–166

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Angle geometry in asymptotic Teichmüller spaces ☆

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ARTICLE INFO

Article history:
Received 26 April 2016
Available online 12 October 2016
Submitted by V. Andrievskii

Keywords:
Teichmüller space
Asymptotic Teichmüller space
Geodesic
Angle

ABSTRACT

The angular geometry of asymptotic Teichmüller spaces is studied. Although it has been proved that the angles between any two geodesic rays from the base point of $AT(X)$ always exist, it is shown in this paper that there are infinitely many pairs of intersecting geodesic segments such that the angles between them do not exist. The sums of inner angles of geodesic triangles are also studied. It is proved that for any number from 0 to 3π there exists a geodesic triangle in $AT(X)$ such that the sum of its inner angles is equal to such a number.

1. Introduction

In this paper, a hyperbolic Riemann surface always means a Riemann surface whose universal covering is conformally equivalent to the unit disk $\mathbb{D}$. Let X be a hyperbolic Riemann surface and let $\mathrm{Belt}(X)$ be the Banach space of all Beltrami differentials $\mu = \mu(z)\overline{dz}/dz$ on X with L_∞-norms. Denote by $M(X)$ the open unit ball of $\mathrm{Belt}(X)$.

For each $\mu \in M(X)$, there is a quasiconformal mapping $f^\mu : X \to f^\mu(X)$ with μ as its complex dilatation. f^μ is uniquely determined up to post-composition a conformal mapping. Two elements μ and ν in $M(X)$ are said to be Teichmüller equivalent, denoted by $\mu \sim \nu$, if there is a conformal mapping $\varphi : f^\mu(X) \to f^\nu(X)$ such that $(f^\nu)^{-1} \circ \varphi \circ f^\mu$ is homotopic to the identity mapping of X modulo the ideal boundary ∂X. The Teichmüller space $T(X)$ of X is the space of all Teichmüller equivalence classes, that is,

$$T(X) := M(X)/\sim = \{[\mu]_T : \ \mu \in M(X)\},$$

where $[\mu]_T$ is the Teichmüller equivalence class containing μ.

☆ This research is partially supported by the National Natural Science Foundation of China (Grant No. 11371045).
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http://dx.doi.org/10.1016/j.jmaa.2016.10.014
0022-247X/

It is known that there is a natural metric on $T(X)$, called Teichmüller metric, such that $T(X)$ is a complete metric space. Furthermore, $T(X)$ is a complex manifold modeled on a complex Banach space such that the quotient mapping $\Phi : M(X) \to T(X)$ is a holomorphic split submersion.

The asymptotic Teichmüller space $AT(X)$ of X is an important relative of the Teichmüller space, which is defined as the space of all asymptotic Teichmüller equivalence classes, that is,

$$AT(X) := M(X)/\sim_{AT} = \{[\mu]_{AT} : \ \mu \in M(X)\},$$

where $[\mu]_{AT}$ is the asymptotic Teichmüller equivalence class containing μ. The asymptotic Teichmüller equivalence $\sim_{AT}$ has the same definition as the Teichmüller equivalence with one exception that the mapping φ is allowed to be asymptotically conformal. A quasiconformal mapping $\varphi : X \to \varphi(X)$ is called asymptotically conformal if, for every $\varepsilon > 0$, there is a compact subset E of X such that the dilatation of φ outside E is less than $1 + \varepsilon$.

$AT(X)$ is of meaning only when X is of infinite analytic type, otherwise it is a single point. In what follows a Riemann surface X is always assumed to be of infinite analytic type.

The asymptotic Teichmüller space was introduced by Gardiner and Sullivan [10] for the unit disk $\mathbb{D}$, and by Earle, Gardiner and Lakic [2,3] for arbitrary hyperbolic Riemann surfaces (see [8] also). It is known that there exists a complex Banach manifold structure and a natural metric, called the Teichmüller metric on $AT(X)$. This metric can be induced by a Finsler form. We refer to [3,6] for more information on $AT(X)$.

The geodesic geometry of Teichmüller metric has been fully studied (see [1,4,5,12,13,16] for example). Recently, an approach to define an angle between two geodesics with the same initial point in a Teichmüller space was given. It was proved in [15] that such angles always exist between any two geodesic rays in infinite dimensional Teichmüller spaces. The existence and the explicit formula of so-defined angle between general geodesic segments were further discussed in [11].

The following problem of angle in Teichmüller spaces was posed in [15].

Problem 1.1. Suppose, there exist three angles of a triangle in $T(X)$, the sides of which are geodesic segments. Is the sum of the three inner angles less than π?

Fan and Jiang [7] gave a negative answer to this problem for the universal Teichmüller space. At the same time, Hu and Shen [11] proved that for any given number from 0 to 3π there exists a geodesic triangle in an arbitrary infinite dimensional Teichmüller space whose inner angles sum is equal to such a number. They also showed that there exist infinitely many pairs of intersecting geodesic segments in $T(X)$ whose included angles do not exist.

In our recent paper [17], we introduced the notion of the angle in asymptotic Teichmüller spaces and studied the existence and explicit formula of so-defined angle. Although it was proved in [17] that angles between any two geodesic rays from the base point of $AT(X)$ always exist, it is shown here that there are infinitely many pairs of intersecting geodesic segments such that the angles between them do not exist.

Theorem 1.1. *Let $AT(X)$ be an asymptotic Teichmüller space of a Riemann surface X of infinite analytic type. There exist infinitely many pairs of intersecting geodesic segments such that the angles between them do not exist.*

Furthermore, we will discuss Problem 1.1 for asymptotic Teichmüller spaces. Similar to that in Teichmüller spaces, we have the following result for asymptotic Teichmüller spaces.

Theorem 1.2. *Let $AT(X)$ be an asymptotic Teichmüller space of a Riemann surface X of infinite analytic type. For any given number $m \in [0, 3\pi]$, there exists a geodesic triangle in $AT(X)$ such that the sum of its three inner angles is equal to m.*