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On the radial solutions of a system with weights under the Keller–Osserman condition

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ABSTRACT

We establish conditions on the functions p_1 and p_2 that are necessary and sufficient for the existence of positive solutions, bounded and unbounded, of the semilinear elliptic system

$$\begin{cases} \Delta u = p_1(|x|) f_1(u,v) & \text{for } x \in \mathbb{R}^N (N \geqslant 3), \\ \Delta v = p_2(|x|) f_2(u,v) & \text{for } x \in \mathbb{R}^N (N \geqslant 3), \end{cases}$$

where p_1, p_2, f_1 and f_2 are continuous functions satisfying certain properties.

1. Introduction

The study of existence of large solutions for semilinear elliptic systems of the form

$$\begin{cases} \Delta u = p_1(|x|) f_1(u,v) & \text{for } x \in \mathbb{R}^N \ (N \geqslant 3), \\ \Delta v = p_2(|x|) f_2(u,v) & \text{for } x \in \mathbb{R}^N \ (N \geqslant 3), \end{cases} \tag{1.1}$$

goes back to the pioneering papers by Keller [7] and Osserman [19]. From the results of [19, Lemma 3, p. 1643] we know that, for a given positive, continuous and nondecreasing function f, the semilinear elliptic partial differential inequality

$$\Delta u \geqslant f(u) \ \text{ in } \mathbb{R}^N, \tag{1.2}$$

possesses a large solution u if and only if the nowadays called Keller–Osserman condition holds, i.e.

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$$KO_f := \int_1^{\infty} \frac{1}{\sqrt{F(s)}} ds = +\infty, \ (F(s) = \int_0^s f(t)\, dt). \tag{1.3}$$

Here, we extend the existence result to the case of systems where the functions p_1 and p_2 are spherically symmetric. More generally, however, we are interested in the influence of the functions p_1 and p_2 on existence results. Here, instead of fixing the conditions on p_1 and p_2, we fix the conditions on f_1 and f_2 and determine sufficient condition on p_1 and p_2 that ensure that (1.1) has an entire solution and whether such solutions are bounded or unbounded and, perhaps, large. Finally, we note that the study of large or bounded solutions for (1.1) when $KO_f < \infty$ or $KO_f = \infty$ has been the subject of many articles. See, for example, the author [3], Nehari [13], Rhee [16] and Redheffer [17], and their references.

The problems of the form (1.1) and (1.2) are drawn by the mathematical modelling of many natural phenomena related to steady-state reaction-diffusion, subsonic fluid flows, electrostatic potential in a shiny metallic body inside or subsonic motion of a gas, automorphic functions theory, geometry and control theory (see, for example, L. Bieberbach [1], Grosse–Martin [5], Diaz [4], Keller [8], Lasry and Lions [10], Matero [12], Pohozaev [14], Rademacher [15] and Smooke [20] for a more detailed discussion). For example, reading the work of Lasry and Lions [10], we can observe that such problems arise in stochastic control theory. The controls are to be designed so that the state of the system is constrained to some region. Finding optimal controls is then shown to be equivalent to finding large solutions for a second order nonlinear elliptic partial differential equation.

2. The main results

Let $a_1, a_2 \in (0,\infty)$ be fixed. We assume:

(P1) $p_1, p_2 : [0,\infty) \to [0,\infty)$ are spherically symmetric continuous functions (i.e., for $r = |x|$ we have $p_1(x) = p_1(r)$ and $p_2(x) = p_2(r)$);

(C1) $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous, increasing in each variables and $f_1(u,v) > 0$, $f_2(u,v) > 0$ whenever $u, v > 0$;

(C2) there exist the continuous and increasing functions $\overline{f_1}, \overline{f_2} : [0,\infty) \to [0,\infty)$, $\bar{c}_1, \bar{c}_2 \in (0,\infty)$, $M_1 \geqslant \max\left\{1, \frac{1}{a_1}\right\}$ and $M_2 \geqslant \max\left\{1, \frac{1}{a_2}\right\}$ such that

$$f_1(t_1, t_1 s_1) \leqslant \bar{c}_1 f_1(t_1, t_1)\, \overline{f_1}(s_1), \ \forall t_1 \geqslant a_1, \ \forall s_1 \geqslant M_1, \tag{2.1}$$

$$f_2(t_2 s_2, t_2) \leqslant \bar{c}_2 f_2(t_2, t_2)\, \overline{f_2}(s_2), \ \forall t_2 \geqslant a_2, \ \forall s_2 \geqslant M_2. \tag{2.2}$$

The assumption (C2) is further discussed in the famous book of Krasnosel'skii and Rutickii [6] (see also Rao and Ren [18]). A simple example of f_1 and f_2 satisfying (C1) and (C2) is given by $f_1(u,v) = u^{\beta_1} v^{\alpha_1}$ and $f_2(u,v) = u^{\beta_2} v^{\alpha_2}$ with $\alpha_i, \beta_i \in [0,\infty)$ such that $\alpha_i^2 + \beta_i^2 \neq 0$ $(i = 1, 2)$.

Next, we introduce the following notations

$$\mathcal{KO}_{f_1}(r) = \int_{a_1}^{r} \frac{1}{\sqrt{\int_0^s f_1(t,t)\, dt}} ds, \ \mathcal{KO}_{f_1}(\infty) = \lim_{s\to\infty} \mathcal{KO}_{f_1}(s),$$

$$\mathcal{KO}_{f_2}(r) = \int_{a_2}^{r} \frac{1}{\sqrt{\int_0^s f_2(t,t)\, dt}} ds, \ \mathcal{KO}_{f_2}(\infty) = \lim_{s\to\infty} \mathcal{KO}_{f_2}(s),$$

$$Z(r) = \int_{a_1+a_2}^{r} \frac{1}{f_1(t,t) + f_2(t,t)} dt, \ \varepsilon_i \in (0,\infty), \ \phi_i(s) = \max_{0\leqslant t\leqslant s} p_i(t), \ i = 1, 2,$$