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An anisotropic Mumford–Shah model

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A B S T R A C T

A variational model is introduced for the segmentation problem of thin structures, like tubes or thin plates, in an image. The energy is based on the Mumford–Shah model with a surfacic term perturbed by a Finsler metric. The formulation in the special space of functions with bounded variations is given and, in order to get an energy more adapted for numerics, a result of Γ-convergence is proved.

1. Introduction

This work is motivated by the segmentation problem of sets strongly elongated in some directions as, for instance, tubes or thin plates in an image of dimension $n \in \{2; 3\}$. In Computer Vision, the Mumford–Shah model is one of the most studied [26]. It consists, for a given image $g \in \mathrm{L}^\infty(\Omega)$, in finding a couple (u, K) which minimizes the following energy

$$\mathcal{E}_{MS}(u, K) = \int_{\Omega\setminus K} (u-g)^2 \mathrm{d}x + \int_{\Omega\setminus K} |\nabla u|^2 \mathrm{d}x + \mathcal{H}^{n-1}(K), \tag{1.1}$$

where $u \in W^{1,2}(\Omega \setminus K)$, K is compact and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. To minimize this energy, K must fit the set of discontinuity of the image and u must represent the regular part of the intensity. In order to adapt this model for the particular case of thin and elongated sets, we have introduced in [27] a Finsler metric φ which must fit the anisotropy of the image. At any point $x \in \Omega$, $\varphi(x, \cdot)$ is a norm whose unit ball coincides with the elongation of the sets we want to detect. So, we set

$$\mathcal{E}(u, K) = \int_{\Omega\setminus K} (u-g)^2 \mathrm{d}x + \int_{\Omega\setminus K} |\nabla u|^2 \mathrm{d}x + \int_K \varphi(x, \nu) \mathrm{d}\mathcal{H}^{n-1}, \tag{1.2}$$

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where $u \in W^{1,2}(\Omega \setminus K)$, K is a compact $(n-1)$-dimensional submanifold and ν is an unit vector orthogonal to K. This kind of model which consists in an energy with a volumic and a surfacic parts also arises in Fracture Mechanics Theory ([11,13,23] for example). In this setting, the analysis for the case where φ is a constant norm has been treated in [21]. To our knowledge, the inhomogeneous case, where φ also depends on x, has not been done yet.

Since the compact submanifolds of Ω cannot be endowed with a topology which ensures that the direct methods apply, a weak formulation of the problem is needed. To do this, De Giorgi and Ambrosio [17] proposed to set this kind of problem in the space SBV of special functions with bounded variation. Thus, setting $K = J_u$ in (1.2) and defining $E(u) = \mathcal{E}(u, J_u)$, it gives

$$E(u) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \varphi(x, \nu_u) \mathrm{d}\mathcal{H}^{n-1}, \tag{1.3}$$

where $u \in \mathrm{SBV}(\Omega)$, ∇u is the derivative of u with respect to the Lebesgue measure, J_u is its jump set and ν_u is an unit vector orthogonal to J_u. The abstract theory in SBV has been developed: Ambrosio established the existence result [1,2], and regularity for minimizers of this kind of energy has been proved [18,4,5,12]. Those results ensure that any minimizer u of the relaxed problem in SBV provides a couple $(u, \overline{J_u})$ which also minimizes the initial model $\mathcal{E}$.

The numerical approximation for solutions is hard because of the treatment of the jump set J_u. To overcome this difficulty, the idea is to perform a variational approximation of the functional E in the sense of De Giorgi Γ-convergence [19,25] with Ambrosio–Tortorelli's approximation.

In order to approximate (1.3), we propose two slightly different families of functionals $(E_\varepsilon)_\varepsilon$ and $(\widetilde{E}_\varepsilon)_\varepsilon$ defined by

$$E_\varepsilon(u,z) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) \mathrm{d}x, \tag{1.4}$$

$$\widetilde{E}_\varepsilon(u,z) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 [\eta_\varepsilon + (1-z^2)^2] \mathrm{d}x + \int_\Omega \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) \mathrm{d}x. \tag{1.5}$$

For both versions, the function z takes its values in $[0;1]$ and plays the role of a control on the gradient of u. In the second one, the parameter η_ε is infinitesimal with respect to ε. The first functional is directly inspired by the initial Ambrosio–Tortorelli's approximation [6], while the second one was introduced later in [7] and it was used in various papers, for example [21,22,15]. Those functionals are more adapted for numerics since usual finite element methods can be directly applied. They formally differ by the introduction of the term

$$\eta_\epsilon \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{1.6}$$

By this way, $\widetilde{E}_\varepsilon$ admits as natural domain of definition the classical Sobolev space $(W^{1,2}(\Omega))^2$. However, this term is not strictly necessary in our study because all the results which are proven for $\widetilde{E}_\varepsilon$ are also true for E_ε. For this simplification, the cost to pay is a slightly longer analysis in order to introduce an adapted domain for E_ε which ensures that its minimization is still a well-posed problem.

The Γ-convergence result when $\varepsilon \to 0^+$ will be proven for both E_ε and $\widetilde{E}_\varepsilon$. However, in the Image Processing context, the parameter ε is devoted to be *small* but *fixed*. Indeed, for stability of the algorithms, ε has to be bounded by below by a positive constant which depends on the size of the grid (see [14]). As in practice we can not take the limit $\varepsilon \to 0^+$, we may choose the first version in order to avoid the additional diffusion in the numerics due to the term (1.6).