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Maps on the positive definite cone of a C^*-algebra preserving certain quasi-entropies

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ABSTRACT

We describe the structure of those bijective maps on the cone of all positive invertible elements of a C^*-algebra with a normalized faithful trace which preserve certain kinds of quasi-entropy. It is shown that essentially any such map is equal to a Jordan *-isomorphism of the underlying algebra multiplied by a central positive invertible element.

1. Introduction

Relative entropy is a numerical quantity which is of fundamental importance in information sciences. It is used to measure dissimilarity between probability distributions in classical information theory, or between quantum states (represented by density operators) in quantum information theory. The original concept of relative entropy was generalized in many different ways in the past decades. In the classical theory, probably the most extensively studied such generalization is the so-called 'f-divergence' introduced by I. Csiszár. As for quantum information science, a natural quantum analogue appeared under the name 'quantum f-divergence', but in fact an even more general concept called 'quasi-entropy' was introduced by D. Petz and then studied in detail.

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Motivated by the classical result of Wigner concerning the structure of quantum mechanical symmetry transformations (i.e., maps on pure states preserving transition probability), in the paper [6] we described the bijective maps of the set of all density operators (positive semidefinite operators with trace 1) on a finite dimensional Hilbert space which preserve the usual quantum relative entropy, i.e., the one due to Umegaki. After this, first in [11] we were able to drop the condition of bijectivity in the former result and then in [9] we could present the complete description of all transformations on the set of density operators which preserve any quantum f-divergence, f being any strictly convex function. (For some more preservers on quantum structures we refer to Chapter 2 in the volume [5].)

For several reasons, in many cases (even in physically motivated ones) transformations defined not only on the set of density operators but on the whole cone of positive (definite or semidefinite) operators are also studied. Especially, when differential geometrical considerations are made and corresponding tools are used, it is very natural to consider the cone of all positive invertible (i.e., positive definite) operators.

As recent literature on investigations in this direction we mention our paper [10] where bijective maps on the cone of positive definite matrices preserving rather general Bregman divergences or Jensen divergences were described and also the paper [18] by Virosztek who managed to determine the structure of all bijective maps on the cone of all positive semidefinite matrices which preserve a general quantum f-divergence.

Our aim here is to make steps towards the descriptions of maps preserving quasi-entropies which concept, as mentioned above, is more general than the one called quantum f-divergence. A closer look at the clever proof given in [18] clearly shows and can convince anybody that the description of general quasi-entropy preservers on cones is most probably a very difficult problem, if it is doable at all. Therefore, in what follows we consider the physically most important quasi-entropies and determine the structures of their preservers.

For these reasons, that is, since here we deal with some particular quasi-entropies, in the original version of the paper we did not give their general definition but referred the reader to the sources [15,13]. We merely recalled that Petz's general quasi-entropies form a large class of numerical quantities of two variables varying over the positive definite cone (i.e., the set of all positive invertible elements) of a finite dimensional C^*-algebra $\mathcal{A}$ which are parametrized by two objects: on the one hand, by a continuous numerical function on the positive real line and, on the other hand, by an element of $\mathcal{A}$. It is written on page 113 in [13] that most of the numerical functions are in fact physically irrelevant, the important ones are just the following: $x \mapsto x \log x$, $x \mapsto -\log x$ and $x \mapsto x^\alpha$ with some exponent $\alpha \in \mathbb{R}$. Nevertheless, following the request by the reviewer of the paper, below we do present the general definition of quasi-entropy at least in the finite dimensional case.

As mentioned above the bijective quantum f-divergence preservers on the cone of all positive semidefinite operators on a finite dimensional Hilbert space have been determined in [18]. As to quasi-entropies, this means the cases where the parametrizing real functions of quasi-entropies under considerations are general but the other parameter, namely the element of the C^*-algebra behind, is the identity. We have already pointed out above that the fully general cases (general real functions and general C^*-algebra elements as parameters) seem at the moment hopeless to handle, therefore we deal with the physically most important concrete cases regarding the function parameter and, on the other hand, we consider the operator parameter rather general (any invertible element). We also mention that although in [15] finite dimensional C^*-algebras were treated, in this paper we do not make restriction on dimension and hence obtain, from the mathematical point of view, general results.

Let us now fix the notation and give some necessary definitions. In what follows let $\mathcal{A}$ be an arbitrary C^*-algebra with unit 1. Denote by $\mathcal{A}_s$ the space of all self-adjoint elements of $\mathcal{A}$, let $\mathcal{A}_+$ stand for the set of all positive elements of $\mathcal{A}$, and denote by $\mathcal{A}_+^{-1}$ the set of all invertible elements of $\mathcal{A}_+$. We call $\mathcal{A}_+^{-1}$ the positive definite cone of $\mathcal{A}$.

By a normalized trace on $\mathcal{A}$ we mean a positive linear functional Tr which satisfies $\operatorname{Tr}(xy) = \operatorname{Tr}(yx)$, $x, y \in \mathcal{A}$, and $\operatorname{Tr} 1 = 1$. It is called faithful if for any $a \in \mathcal{A}_+$, the equality $\operatorname{Tr} a = 0$ implies $a = 0$.