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On maximal relative projection constants

Simon Foucart [a,*,1], Lesław Skrzypek [b]

[a] Texas A&M University, United States
[b] University of South Florida, United States

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ABSTRACT

This article focuses on the maximum of relative projection constants over all m-dimensional subspaces of the N-dimensional coordinate space equipped with the max-norm. This quantity, called maximal relative projection constant, is studied in parallel with a lower bound, dubbed quasimaximal relative projection constant. Exploiting alternative expressions for these quantities, we show how they can be computed when N is small and how to reverse the Kadec–Snobar inequality when N does not tend to infinity. Precisely, we first prove that the (quasi)maximal relative projection constant can be lower-bounded by $c\sqrt{m}$, with c arbitrarily close to one, when N is superlinear in m. The main ingredient is a connection with equiangular tight frames. By using the semicircle law, we then prove that the lower bound $c\sqrt{m}$ holds with $c < 1$ when N is linear in m.

1. Introduction

This article investigates relative projection constants of m-dimensional subspaces $\mathcal{V}_m$ of ℓ_N^∞, which are defined as

$$\lambda(\mathcal{V}_m, \ell_\infty^N) := \min\left\{ |||P|||_{\infty\to\infty} : \ P \text{ is a projection from } \ell_\infty^N \text{ onto } \mathcal{V}_m \right\}, \tag{1}$$

and more specifically maximal relative projection constants, which are defined as

$$\lambda(m, N) := \max\left\{ \lambda(\mathcal{V}_m, \ell_\infty^N) : \ \mathcal{V}_m \text{ is an } m\text{-dimensional subspace of } \ell_\infty^N \right\}. \tag{2}$$

With $\mathbb{K}$ denoting either $\mathbb{R}$ or $\mathbb{C}$, we append a subscript $\mathbb{K}$ in the notation $\lambda_{\mathbb{K}}(m, N)$ to indicate that $\ell_\infty^N = (\mathbb{K}^N, \|\cdot\|_\infty)$ is understood as a real or a complex linear space. The existing literature often deals with maximal absolute projection constants, which may be defined as

* Corresponding author.
E-mail address: foucart@tamu.edu (S. Foucart).

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$$\lambda_{\mathbb{K}}(m) := \sup_{N \geq m} \lambda_{\mathbb{K}}(m, N).$$

As a representative example, the Kadec–Snobar estimate $\lambda_{\mathbb{K}}(m) \leq \sqrt{m}$ can be proved by several different approaches — we can add yet another approach based on Theorem 1, see Remark 10. Following earlier works such as [12] and [5], we focus on the properties of $\lambda_{\mathbb{K}}(m,N)$ with N fixed rather than the properties of $\lambda_{\mathbb{K}}(m)$. In particular, we are looking at reversing the Kadec–Snobar inequality with N moderately large. In Section 2, we highlight two alternative expressions for the maximal relative projection constant $\lambda_{\mathbb{K}}(m,N)$. They are only new in the case $\mathbb{K} = \mathbb{C}$, but even in the case $\mathbb{K} = \mathbb{R}$, the arguments we propose contrast with the ones found in the literature. We also introduce a related quantity $\mu_{\mathbb{K}}(m,N)$, dubbed quasimaximal relative projection constant, which is a lower bound for $\lambda_{\mathbb{K}}(m,N)$. We then establish some common properties shared by $\lambda_{\mathbb{K}}(m,N)$ and $\mu_{\mathbb{K}}(m,N)$. In Section 3, we focus on the computation of these quantities. We show how to determine (for small N) the exact value of $\mu_{\mathbb{R}}(m,N)$ and the value of a lower bound for $\lambda_{\mathbb{R}}(m,N)$, which is in fact believed to be the true value. In particular, we reveal that $\lambda_{\mathbb{R}}(m,N)$ and $\mu_{\mathbb{R}}(m,N)$ really do differ in general. In Section 4, we make explicit a connection between equiangular tight frames and specific values for $\lambda_{\mathbb{K}}(m,N)$ and $\mu_{\mathbb{K}}(m,N)$. Based on these considerations, we prove that the Kadec–Snobar estimate is optimal in the sense that there are spaces $\mathcal{V}_m$ of arbitrarily large dimension m such that $\lambda_{\mathbb{K}}(m)/\sqrt{m}$ (or in fact $\mu_{\mathbb{K}}(m)/\sqrt{m}$) is arbitrarily close to one. This is only new in the case $\mathbb{K} = \mathbb{R}$. However, in the examples provided in Section 4, the dimension N of the superspace grows superlinearly in m. To the best of our knowledge, such a result was previously achieved only with N growing quadratically in m. In Section 5, we further show that a lower estimate $\lambda_{\mathbb{K}}(m,N) \geq c\sqrt{m}$ (or in fact $\mu_{\mathbb{K}}(m,N) \geq c\sqrt{m}$) is actually possible with N growing only linearly in m. For this, we rely on the alternative expression for $\lambda_{\mathbb{K}}(m,N)$ in terms of eigenvalues of Seidel matrices and invoke the semicircle law for such matrices chosen at random. We conclude the article with some remarks linking minimal projections to matrix theory and graph theory via the alternative expression for the maximal relative projection constant $\lambda_{\mathbb{K}}(m,N)$ highlighted at the beginning. Four appendices collect some material whose inclusion in the main text would have disrupted the flow of reading.

Notation: The blackboard-bold letter $\mathbb{K}$ represents either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. The notation $\mathbb{R}^n_+$ stands for the set of vectors with n nonnegative real entries, just like $\mathbb{K}^n$ stands for the set of vectors with n entries in $\mathbb{K}$. As a linear space, the latter may be equipped with the usual p-norm $\|\cdot\|_p$ for any $p \in [1,\infty]$, in which case it is represented by ℓ^n_p. Given a vector $v \in \mathbb{K}^n$, the notation $\mathrm{diag}(v)$ refers to the diagonal matrix in $\mathbb{K}^{n\times n}$ with v on its diagonal. The modulus (or absolute value) $|M|$ of a matrix $M \in \mathbb{K}^{n\times n}$ is understood componentwise, so that its (i,j)th entry is $|M|_{i,j} = |M_{i,j}|$. The adjoint of a matrix $M \in \mathbb{K}^{n\times n}$ is the matrix M^* with (i,j)th entry $M^*_{i,j} = \overline{M_{j,i}}$. The eigenvalues $\lambda^{\downarrow}_1(M), \lambda^{\downarrow}_2(M), \ldots, \lambda^{\downarrow}_n(M)$ of a self-adjoint matrix $M \in \mathbb{K}^{n\times n}$ are arranged in nonincreasing order, so that $\lambda^{\downarrow}_1(M) \geq \lambda^{\downarrow}_2(M) \geq \cdots \geq \lambda^{\downarrow}_n(M)$. The squared Frobenius norm $\|M\|^2_F = \sum_{i,j=1}^n |M_{i,j}|^2$ of a matrix $M \in \mathbb{K}^{n\times n}$ can also be written as $\|M\|^2_F = \sum_{k=1}^n \lambda^{\downarrow}_k(M)^2$. We use the letter B to represent a Seidel matrix, i.e., a self-adjoint matrix $B \in \mathbb{K}^{n\times n}$ with $B_{i,i} = 0$ for all $i \in [\![1:n]\!]$ and $|B_{i,j}| = 1$ for all $i \neq j \in [\![1:n]\!]$ — in the case $\mathbb{K} = \mathbb{R}$, these matrices are often called Seidel adjacency matrices. The set of $n \times n$ Seidel matrices is denoted by $\mathcal{S}^{n\times n}_{\mathbb{K}}$. We use the letter A to represent a matrix of the form $A = I_n + B$ where $B \in \mathcal{S}^{n\times n}_{\mathbb{K}}$, i.e., a self-adjoint matrix with the diagonal entries equal to one and off-diagonal entries having a modulus (or absolute value) equal to one.

2. Conversion of maximal relative projection constants

In this section, we highlight two alternative expressions for $\lambda_{\mathbb{K}}(m,N)$ that turn out to be useful for establishing some properties of the maximal relative projection constants, e.g. the properties listed in Proposition 2 below. These expressions are not new: the $\leq$-part of (3) uses trace duality and dates back to [13]