J. Math. Anal. Appl. 447 (2017) 329–353

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fundamental solutions of the time fractional diffusion-wave and parabolic Dirac operators

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ARTICLE INFO

Article history:
Received 27 May 2016
Available online 11 October 2016
Submitted by J. Xiao

Keywords:
Time fractional diffusion-wave operator
Time fractional parabolic Dirac operator
Fundamental solutions
Caputo fractional derivative
Fractional moments

ABSTRACT

In this paper we study the multidimensional time fractional diffusion-wave equation where the time fractional derivative is in the Caputo sense with order $\beta \in]0,2]$. Applying operational techniques via Fourier and Mellin transforms we obtain an integral representation of the fundamental solution (FS) of the time fractional diffusion-wave operator. Series representations of the FS are explicitly obtained for any dimension. From these we derive the FS for the time fractional parabolic Dirac operator in the form of integral and series representation. Fractional moments of arbitrary order $\gamma > 0$ are also computed. To illustrate our results we present and discuss some plots of the FS for some particular values of the dimension and of the fractional parameter.

1. Introduction

Fractional diffusion-wave equations are obtained from the standard diffusion and wave equations by replacing time and/or space derivatives by a fractional derivative of order $\beta \in]0,2]$, for example, Riemann–Liouville, Weyl, Caputo, Riesz, or Riesz–Feller, just to mention some of the most used types of fractional derivatives. The introduction of fractional derivatives allows to represent the physical reality more accurately by introducing a memory mechanism in the process (see [2]). These equations represent anomalous diffusion ($0 < \beta < 1$) or anomalous wave propagation ($1 < \beta < 2$) and have been studied over the last years by several authors. Just to clarify, an anomalous diffusion propagation process corresponds to a propagation process that does not follow Gaussian statistics on long time intervals.

In the one dimensional case, solutions of time and/or space fractional diffusion-wave equations have been constructed and studied comprehensively in several papers (see, for example, [7,20,22–26,30,34] and the

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http://dx.doi.org/10.1016/j.jmaa.2016.08.052

references therein indicated). The fundamental solution (FS) of the time fractional diffusion-wave equation represents a slow diffusion process for $0 < \beta < 1$ whereas for $1 < \beta < 2$ it represents a wave diffusion faster than the Gaussian diffusion. This gives a unification of diffusion and wave propagation phenomena. One of the first works in this direction was made by Wyss in [34]. Here the author obtained the FS of the one dimensional time fractional diffusion equation in the form of Fox H-functions. In [7] the author studied independently the fractional diffusion-wave equation, obtaining not only a representation for the FS but also additional properties of it. In [23] the FS for the Cauchy and Signalling problems associated to the time fractional diffusion-wave equation were expressed in terms of entire functions of Wright type. In [26] it was showed that the FS obtained in [23] can be interpreted as a spatial probability density function evolving in time with similarity properties. In [20,22,24,25] the authors studied and obtained the FS for more general equations where both space and time derivatives are fractional.

For the multidimensional case there are some works in this direction (see e.g. [11,12,21,30]). In [30] a closed form of the FS in terms of Fox H-functions was obtained and some of their properties were studied. In [11,12, 21] multidimensional time-fractional and space–time fractional diffusion-wave equations were investigated. However, in these works a generic series representation for the FS in an arbitrary dimension was not obtained. In [11,12] there are only integral representations for the FS and in [21] series representations were obtained up to dimension 3 for the neutral fractional wave equation. Furthermore, a representation of the FS in the form of an absolute convergent series enables us to handle these functions in an easier way and to apply them in approximations.

In this paper we obtain explicitly integral and series representations for the FS of the diffusion-wave equation, for an arbitrary dimension. The computations are much more involved and the series representations obtained depend on the parity of the dimension. In connection with the time fractional diffusion-wave operator we also consider the time fractional parabolic Dirac operator. This is a first-order differential operator in space combined with a fractional derivative of order $\beta \in]0, 2]$ in time, written using a Witt basis. This operator factorizes the time fractional diffusion-wave operator. Hence, its solutions can be seen as a refinement of the solutions of the time fractional diffusion-wave operator. We also obtain the FS of the time fractional parabolic Dirac operator for an arbitrary dimension. This opens new possibilities for the development of a fractional function theory for this operator in the context of Clifford analysis and the study, e.g., of the fractional Schrödinger equation. For the integer case $\beta = 1$ the parabolic Dirac operator was proposed in [4] using a Clifford algebra approach to study the time-dependent Navier–Stokes equation. This allowed a successful adaptation of already existent techniques in elliptic function theory (see [10]) to non-stationary problems in time-varying domains (see for example [3,17,32]). The geometric nature of Clifford algebras allows the resolution of PDEs using the geometric properties of the domain where the differential operator acts (see [4]). Connections between Clifford analysis and fractional calculus were recently established in [6,14,33] in the study of the stationary fractional Dirac operator.

The structure of the papers reads as follows: in the preliminaries section we recall some basic concepts about Clifford analysis, Witt basis, fractional calculus, special functions and integral transforms. In Sections 3 and 4 we construct integral and series representations for the FS of the time fractional diffusion-wave operator and the time fractional parabolic Dirac operator in $\mathbb{R}^n \times \mathbb{R}^+$, respectively. These representations depend on the parity of the space dimension. The particular cases of $\beta = 0, 1, 2$ are discussed in Section 5. Fractional moments of arbitrary order $\gamma > 0$ are computed in Section 6 for the FS of the time fractional diffusion-wave operator. Finally, in Sections 7 and 8 we present and discuss some plots of the FS obtained in Sections 3 and 4.