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The eigenvalue problem and infinitely many sign-changing solutions for an elliptic equation with critical Hardy constant ☆

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ABSTRACT

By Li–Yau's idea and a more precise Hardy's inequality, we obtain a new lower bound of eigenvalues for an elliptic equation with critical Hardy constant. Then using the lower bounds, we prove that there are infinitely many sign-changing solutions for a class of elliptic equation with critical Hardy constant which extends the results in Schechter and Zou (2005) [9] to the critical Hardy constant case.

1. Introduction and main results

We first consider the lower bounds of eigenvalues for the following eigenvalue problem:

$$\begin{cases} -\triangle u - \frac{(n-2)^2}{4}\frac{u}{|x|^2} = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is an open bounded domain with smooth boundary containing the origin.

By the classical Hardy's inequality in [7]:

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(n-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx,$$

we know that the operator $-\triangle - \gamma/|x|^2$ is a positive operator and equivalent to $-\triangle$ when $0 < \gamma < (n-2)^2/4$. Also, in [9] the authors established the lower bounds of eigenvalues as follows:

$$\lambda_k \geq C_0 k^{2/n}, \text{ for some } C_0 > 0 \text{ and } k \geq 1.$$

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In fact, we can also prove that $\lambda_k \le C_1 k^{2/n}$. However, the operator $-\triangle - (n-2)^2/(4|x|^2)$ is not equivalent to $-\triangle$ any more. Inspired by Li–Yau's idea in [8] and a more precise Hardy's inequality in [1], we have

Theorem 1.1. *Let λ_k be the k-th eigenvalue of (1.1), then*

$$\lambda_k \ge C_2 k^{\frac{2}{n}-\frac{2-q}{q}},$$

for any $k \ge 1$ and $q \in (\frac{2n}{n+2}, 2)$, where C_2 is the constant in (2.8) below.

Remark 1.1. The results in Theorem 1.1 show that the power of lower bounds of eigenvalues to (1.1) can be close to $2/n$ arbitrarily by taking q suitably.

Next, we concern with the multiplicity of sign-changing solution for the following nonlinear elliptic equation with critical Hardy constant,

$$\begin{cases} -\triangle u - \frac{(n-2)^2}{4}\frac{u}{|x|^2} = a(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ is an open bounded domain with smooth boundary containing the origin.

Lots of mathematicians have researched the elliptic equation with Hardy potential

$$\begin{cases} -\triangle u - \gamma\frac{u}{|x|^2} = a(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

For the case $0 < \gamma < (n-2)^2/4$, by the asymptotic behavior of the eigenfunctions, Cao–Han [2] and Cao–Peng [3] obtained the existence of multiple solutions and sign-changing solution when $a(x,u)$ is the critical nonlinear term and $\gamma < \bar{\gamma}$, where $\bar{\gamma}$ is a fixed positive number which is smaller than $(n-2)^2/4$. Later, by using the lower bounds of eigenvalues, Schechter–Zou [9] proved that there are infinitely many sign-changing solutions under some conditions on $a(x,u)$, where $a(x,u)$ is subcritical and non-symmetry.

Here we assume that $a(x,u) = f(x,u) + g(x,u)$ satisfies

(H-1) $f\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth:

$$|f(x,u)| \le c(1 + |u|^{p-1}), \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where $p \in (2, 2^*)$, $2^* = 2n/(n-2)$. Moreover, $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \Omega$; and $f(x,u)u \ge 0$ for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$.

(H-2) There exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x,u) \le uf(x,u), \quad x \in \Omega, \ |u| \ge R,$$

where $F(x,u) = \int_0^u f(x,v)dv$.

(H-3) $f(x,u)$ is odd in u, which is $f(x,-u) = -f(x,u)$.

(H-4) $g\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. There exists $\sigma < \min\{\mu/2, p\}$ such that

$$|g(x,u)| \le c(1 + |u|^{\sigma}), \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Moreover, $g(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \Omega$; and $g(x,u)u > 0$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}\backslash\{0\}$.

In [9], if $0 < \gamma < (n-2)^2/4$, the authors proved that the following equation with non-symmetry term